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Analytical solutions of the fractional coupled Konopelchenko-Dubrovsky equation via the modified (w/g)-expansion method

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A B S T R A C T

This study investigates solutions to the fractional (2+1)-dimensional coupled Konopelchenko-Dubrovsky (FKD) equation using the beta fractional derivative method. The main goal is to find exact analytical solutions by applying the modified (w/g)-expansion technique. Several types of solutions with unknown parameters are obtained. To illustrate the results, graphs based on selected parameter values are provided. The results confirm that the modified (w/g)-expansion method is an effective and reliable tool for solving the fractional FKD equation.

1. Introduction

Nonlinear partial differential equations (NLPDEs) are crucial for modeling complex phenomena across various disciplines. They have been extensively employed to represent sophisticated systems, with applications spanning fluid mechanics, material science, environmental studies, biomedical engineering, physics-informed artificial intelligence, and developments in quantum theory (Debnath, 2012; Wazwaz, 2009; He et al., 2024). Various techniques have been employed to obtain exact solutions to NLPDEs, including the Jacobi elliptic function expansion method (Chen and Wang, 2005), (G'/G)−expansion method (Hassaballa et al., 2024), Tanh-function method (Malfliet, 1992), and the modified (w/g)-expansion method (Wen-An et al., 2009). This study employs the modified (w/g)-expansion method to obtain exact analytical solutions for the fractional Konopelchenko-Dubrovsky (KD) equation. The coupled (2+1)-dimensional KD equation, as described by Konopelchenko and Dubrovsky (1984), is mathematically represented as:

$$\frac{\partial v}{\partial t}-\frac{\partial^3 v}{\partial x^3}-6nv\frac{\partial v}{\partial x}+\frac{3}{2}m^2v^2\frac{\partial v}{\partial x}-3\frac{\partial u}{\partial y}+3mu\frac{\partial v}{\partial x}=0, \quad (1)$$

$$\frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}, \quad (2)$$

where, v and u are functions in x, y (spatial variables) and t (time). Additionally, m and n are real-valued parameters. In space-time variables, the coupled fractional KD (FKD) equation is formulated as (Zheng and Feng, 2014):

$${}_0^AD_t^\beta v={}_0^AD_{xxx}^{\beta\beta\beta}v-6nv\,{}_0^AD_x^\beta v+\frac{3}{2}m^2v^2\,{}_0^AD_x^\beta v-3\,{}_0^AD_y^\beta u+3mu\,{}_0^AD_x^\beta v, \quad (3)$$

$${}_0^AD_y^\beta v={}_0^AD_x^\beta u. \quad (4)$$

The fractional coupled FKD equation is relevant in studying nonlinear waves, particularly solitons and integrable systems. The equation can also be used to model the behaviors of light in optical Fibers. In superfluidity, the fractional coupled (2+1)-dimensional FKD can be used to model the behaviors of superfluids, such as helium. The solutions of the equation can describe the propagation of vortices and solitons in these systems. In biology, the FKD equation can model processes such as signal transmission in cellular networks or wave propagation in neural activity (Aslam et al., 2023; Wang and Li, 2024). This study focuses on obtaining exact wave solutions for Eq. 3 and Eq. 4 using the

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beta fractional derivative (BFD). First proposed by Atangana and Goufo (2014), the BFD represents a groundbreaking advancement in fractional calculus, offering key properties that have far-reaching applications across various fields, including mathematics, engineering, and physics. In Eq. 3 and Eq. 4, the BFD concerning t, x, and y is denoted by ${}_0^AD_t^\beta$, ${}_0^AD_x^\beta$ and ${}_0^AD_y^\beta$, respectively. Furthermore, higher-order operations are defined as ${}_0^AD_{xx}^{\beta\beta} v = {}_0^AD_x^\beta({}_0^AD_x^\beta v)$ for the second-order and ${}_0^AD_{xxx}^{\beta\beta\beta} v = {}_0^AD_x^\beta({}_0^AD_{xx}^{\beta\beta} v)$ for the third-order BFDs. BFD is recognized for its properties in effectively characterizing soliton wave behaviors, making it highly suitable for finding solitary solutions. Its advantage lies in its capacity to generate exact solutions while providing profound physical insights into the underlying dynamics. This study takes advantage of these properties by applying the modified (w/g)-expansion method to derive traveling wave solutions for Eq. 3 and Eq. 4, thereby offering a deeper understanding of the wave phenomena described by these equations.

The application of the modified (w/g)-expansion method in deriving exact solutions for NPDEs has been explored in studies such as Gepreel (2016; 2020), while its fractional version has been studied by Deniz et al. (2024). Consequently, its application in fractional NDPES has not been extensively researched. This study addresses this gap by applying the modified (w/g)-expansion method precisely to the FKD equation, a context where this method has not been widely utilized. This approach primarily aims to obtain analytical solutions for the FKD equation. The paper is organized as follows: Section 2 covers the properties of BFD. Section 3 introduces the modified (w/g)-expansion method, while Section 4 applies this method to obtain solutions for the FKD equation. Section 5 presents graphical representations illustrating the physical properties of the derived solutions. Finally, Section 6 concludes the study and offers recommendations for future research.

2. The beta fractional derivative (BFD)

The beta fractional derivative presents distinct benefits compared to traditional fractional derivatives, such as Caputo and Riemann-Liouville. Its extra parameters enhance the ability to model memory and hereditary traits of complex systems. Unlike Caputo, which mandates integer-order initial conditions, or Riemann-Liouville, which can result in unrealistic initial conditions, the beta derivative facilitates more natural and physically relevant formulations. Additionally, it improves both analytical and numerical handling in specific scenarios, making it a practical option for real-world use in areas like anomalous diffusion, viscoelasticity, and control theory (Wang et al., 2022; Nadeem et al., 2024).

According to Atangana and Goufo (2014), the BFD of a function $v(x)$ is defined as:

$$ {}_0^AD_x^\beta v(x) = \lim_{\epsilon\to 0}\frac{v\left(x+\epsilon\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-v(x)}{\epsilon},\ x>0, \beta\in(0,1]. $$

This definition broadens the idea of differentiation to include non-integer orders, allowing it to describe the behavior of functions over a fractional range. Now, suppose $v(x)$ and $g(x)$ are β-differentiable functions, $\forall\,\beta\in(0,1]$, then the following properties are satisfied:

- ${}_0^AD_x^\beta\{b_0 v(x)+b_1 g(x)\} = b_0\,{}_0^AD_x^\beta(v(x)) + b_1\,{}_0^AD_x^\beta(g(x)), \forall\, b_0, b_1\in\mathbb{R}$.
- ${}_0^AD_x^\beta(b_0) = 0$, where b_0 is constant.
- ${}_0^AD_x^\beta(v(x)g(x)) = g(x)\,{}_0^AD_x^\beta(v(x)) + v(x)\,{}_0^AD_x^\beta(g(x))$.
- ${}_0^AD_x^\beta\left(\frac{v(x)}{g(x)}\right) = \frac{g(x)\,{}_0^AD_x^\beta(v(x)) - v(x)\,{}_0^AD_x^\beta(g(x))}{g(x)^2}$.

One significant benefit of the Beta derivative is that it follows the basic principles of classical calculus, such as the product, quotient, and chain rules (Atangana and Goufo, 2014).

3. Description of the modified (w/g)-expansion method

In the following, the key steps of the (w/g)-expansion method are outlined:

- **Step 1:** The nonlinear fractional partial differential equation involving the variables x, y, and t is assumed to be of the form:

$$ F(v, {}_0^AD_t^\beta v, {}_0^AD_x^\beta v, {}_0^AD_y^\beta v, {}_0^AD_{tt}^{\beta\beta} v, {}_0^AD_{tx}^{\beta\beta} v, {}_0^AD_{ty}^{\beta\beta} v, {}_0^AD_{xy}^{\beta\beta} v, {}_0^AD_{xx}^{\beta\beta} v, {}_0^AD_{yy}^{\beta\beta} v, \ldots.) = 0, \tag{5} $$

where, $v(x,y,t)$ and F is a polynomial in v and its various partial derivatives.

- **Step 2:** Solutions to Eq. 5 are obtained by considering the following traveling wave transformation:

$$ v = v(\xi), \qquad \xi = x + y - kt, \tag{6} $$

where, k is a constant. Substituting Eq. 6 into Eq. 5 yields:

$$ F(v, v', v'', \ldots.) = 0. \tag{7} $$

Eq. 7 can be expressed as a polynomial in the form of:

$$ v(\xi) = \sum_{i=-N}^{N} a_i\left(\frac{w}{g}\right)^i, \tag{8} $$

where, a_i $(i = 0, \pm1, \pm2, \dots, \pm N)$ are arbitrary constants and $w(\xi), g(\xi)$ satisfy the following relation:

$$(w/g)' = a + b\left(\frac{w}{g}\right) + c\left(\frac{w}{g}\right)^2. \quad (9)$$

Thus,

$$w'g - wg' = ag^2 + bwg + cw^2, \quad (10)$$

where, a, b, c are arbitrary constants.

- **Step 3:** The positive integer N is determined from Eq. 8 by matching the highest power of $(w/g)^i$ found in the nonlinear terms with the highest power of $(w/g)^i$ in the highest-order derivatives of Eq. 7.
- **Step 4:** Eq. 8 is substituted into Eq. 7, along with Eq. 9. The terms are then grouped according to identical powers of $(w/g)^i$, where $i = 0, \pm1, \pm2, \dots, \pm N$. The coefficients of each power of $(w/g)^i$ are set to zero, resulting in a system of algebraic equations for a_i.
- **Step 5:** The resulting overdetermined system of nonlinear algebraic equations is solved using computational tools such as Maple to determine a_i.
- **Step 6:** Solutions obtained from the previous steps are then used to construct a series of fundamental solutions for Eq. 5.

By selecting $w = gg'$, variations of the g'-expansion method can be derived, leading to an exact solution expressed in the modified form:

$$v(\xi) = \sum_{i=-N}^{N} a_i\,(g')^i. \quad (11)$$

where, g satisfies the nonlinear second-order ordinary differential equation:

$$g'' = a + bg' + c(g')^2, \quad (12)$$

where, $a, b,$ and c are arbitrary constants.

Based on the general solution of Eq. 12, the exact solutions can be classified into the following three families:

- **Family 1:** If $\Omega = 4ac - b^2 > 0$, the solutions are;

$$g(\xi) = \frac{1}{2c}\left[ln\left(\ln(1 + tan^2\left(\frac{1}{2}\sqrt{\Omega}\xi\right)\right) - b\xi\right]. \quad (13)$$

$$g' = \frac{1}{2c}\left[\sqrt{\Omega}\tan(\frac{1}{2}\sqrt{\Omega}\xi) - b\right]. \quad (14)$$

- **Family 2:** If $\Omega = 4ac - b^2 < 0$, the solutions are;

$$g(\xi) = \frac{1}{2c}\left[\ln\left(\tanh^2\left(\frac{1}{2}\sqrt{-\Omega}\,\xi\right) - 1\right) - b\xi\right]. \quad (15)$$

$$g' = -\frac{1}{2c}\left[\sqrt{-\Omega}\tanh(\frac{1}{2}\sqrt{-\Omega}\xi) + b\right]. \quad (16)$$

- **Family 3:** If $\Omega = 4ac - b^2 = 0$, the solutions are;

$$g(\xi) = -\frac{1}{c}\left[\ln(\xi) + \frac{b}{2}\xi\right]. \quad (17)$$

$$g' = -\frac{1}{c}\left[\frac{1}{\xi} + \frac{b}{2}\right]. \quad (18)$$

4. Solutions to the coupled FKD equations

The modified (w/g)-expansion method is applied to construct the exact solution for the FKD equation based on Eq. 3 and Eq. 4. Consider the following traveling wave transformations $v(x, y, t) = v(\xi)$ and $u(x, y, t) = u(\xi)$, where,

$$\xi = \frac{\rho}{\beta}(x + \frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y + \frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t + \frac{1}{\Gamma(\beta)})^\beta.$$

here, ρ, μ, λ are constants and β represents the beta fractional order.

Applying the above transformations to Eqs. 3-4 results in the following equations:

$$-\lambda v' - \rho^3 v''' - 6\rho n v v' + \frac{3}{2}m^2\rho v^2 v' - 3\mu u' + 3m\rho u v' = 0. \quad (19)$$

$$\mu v' = \rho u'. \quad (20)$$

Integrating Eq. 20 with respect to ξ and setting the constant of integration to zero yields;

$\mu v = \rho u$, leading to

$$u = \frac{\mu}{\rho}v. \quad (21)$$

Substituting Eq. 21 into Eq. 19 gives:

$$-\lambda v' - \rho^3 v''' - 6\rho n v v' + \frac{3}{2}m^2\rho v^2 v' - 3\frac{\mu^2}{\rho}v' + 3m\mu v v' = 0. \quad (22)$$

Integrating Eq. 22 results in:

$$-(\lambda + 3\frac{\mu^2}{\rho})v + \frac{3}{2}(m\mu - 6\rho n)v^2 - \rho^3 v'' + \frac{1}{2}m^2\rho v^3 = 0. \quad (23)$$

Applying the balance principle between the terms $v^3$and v'' in Eq. 23 determines $N = 1$. Thus, the solution of the FKD equation takes the form

$$v(\xi) = \sum_{i=-1}^{1} a_i\,(g')^i = a_{-1}(g')^{-1} + a_0 + a_1 g'. \quad (24)$$

Substituting Eq. 24 with Eq. 10 in Eq. 23 and setting the coefficients of g' to zero results in a system of algebraic equations. This leads to a system of nonlinear algebraic equations for $a_{-1}, a_0, a_1, \rho, \mu\ \lambda, a, b$ and c which can be solved using Maple. The resulting sets of equations are as follows:

- **Set 1:** $a_0 = \frac{\rho b}{m}, a_{-1} = \frac{2\rho a}{m}, \mu = \frac{6\rho n}{m},\ \lambda = \frac{\rho(m^2\rho^2b^2 - 4m^2\rho^2ac - 216n^2)}{2m^2}, a_1 = 0, \rho = \rho$ (25)
- **Set 2:** $a_0 = \frac{-\rho b}{m}, a_{-1} = \frac{-2\rho a}{m},\ \mu = \frac{6\rho n}{m},\ \lambda = \frac{\rho(m^2\rho^2b^2 - 4m^2\rho^2ac - 216n^2)}{2m^2}, a_1 = 0, \rho = \rho.$ (26)
- **Set 3:** $a_0 = \frac{\rho b}{m},\ \mu = \frac{6\rho n}{m},\ \lambda = \frac{\rho(m^2\rho^2b^2 - 4m^2\rho^2ac - 216n^2)}{2m^2}, a_1 = \frac{2c\rho}{m}, a_{-1} = 0, \rho = \rho.$ (27)
- **Set 4:** $a_0 = \frac{-\rho b}{m},\ \mu = \frac{6\rho n}{m},\ \lambda = \frac{\rho(m^2\rho^2b^2 - 4m^2\rho^2ac - 216n^2)}{2m^2}, a_1 = \frac{-2c\rho}{m}, a_{-1} = 0, \rho = \rho$ (28)
- **Set 5:** $a_0 = \frac{2\rho b}{m}, \mu = \frac{(6\rho n - \rho^2 mb)}{m},\ \ \lambda = \frac{(-4m^2\rho^3b^2 + 36\rho^2bn + 4m^2\rho^3ac - 108n^2\rho)}{m^2}, a_{-1} = \frac{2\rho a}{m},$ $a_1 = \frac{2c\rho}{m}, \rho = \rho\ .$ (29)

- **Set 6:** $a_0 = \frac{-2\rho b}{m}, \mu = \frac{(6\rho n+\rho^2 mb)}{m}$, $\lambda = \frac{(-4m^2\rho^3 b^2-36\rho^2 bmn+4m^2\rho^3 ac-108n^2\rho)}{m^2}$, $a_{-1} = \frac{-2\rho a}{m}$, $a_1 = \frac{-2c\rho}{m}$, $\rho = \rho$. (30)

- **Family 1:** For $\Omega = 4ac - b^2 > 0$, the solutions in Eq. 24 take the form:

$$v(\xi) = a_{-1}\left(\frac{1}{2c}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\xi\right) - b\right]\right)^{-1} + a_0 + a_1\left(\frac{1}{2c}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\xi\right) - b\right]\right), u(\xi) = \frac{\mu}{\rho}v. \quad (31)$$

Based on the above sets, the following solutions are obtained:

For set 1:

$$\begin{cases} v_1(x,y,t) = \frac{\rho b}{m} + \frac{4\rho ac}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right]^{-1}, \\ u_1(x,y,t) = \frac{b\,\mu}{m} + \frac{4ac\mu}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right]^{-1}, \\ \mu = \frac{6\rho n}{m},\ \lambda = \frac{\rho(m^2\rho^2 b^2-4m^2\rho^2 ac-216n^2)}{2m^2}. \end{cases} \quad (32)$$

For set 2:

$$\begin{cases} v_2(x,y,t) = \frac{-\rho b}{m} - \frac{4\rho ac}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right]^{-1}, \\ u_2(x,y,t) = \frac{-\mu b}{m} - \frac{4ac\mu}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right]^{-1}, \\ \mu = \frac{6\rho n}{m},\ \lambda = \frac{\rho(m^2\rho^2 b^2-4m^2\rho^2 ac-216n^2)}{2m^2}. \end{cases} \quad (33)$$

For set 3:

$$\begin{cases} v_3(x,y,t) = \frac{\rho b}{m} + \frac{\rho}{m}\left(\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right]\right), \\ u_3(x,y,t) = \frac{6nb\rho}{m^2} + \frac{4\mu ac}{m}\left(\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right]\right), \\ \mu = \frac{6\rho n}{m},\ \lambda = \frac{\rho(m^2\rho^2 b^2-4m^2\rho^2 ac-216n^2)}{2m^2}. \end{cases} \quad (34)$$

For set 4:

$$\begin{cases} v_4(x,y,t) = \frac{-\rho b}{m} - \frac{\rho}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right], \\ u_4(x,y,t) = \frac{-\mu b}{m} - \frac{\mu}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right], \\ \mu = \frac{6\rho n}{m},\ \lambda = \frac{\rho(m^2\rho^2 b^2-4m^2\rho^2 ac-216n^2)}{2m^2}. \end{cases} \quad (35)$$

For set 5:

$$\begin{cases} v_5(x,y,t) = \frac{2\rho b}{m} + \frac{4\rho ac}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right]^{-1} + \\ \quad \frac{\rho}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right], \\ u_5(x,y,t) = \frac{2\mu b}{m} + \frac{4\mu ac}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right]^{-1} \\ \quad + \frac{\mu}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right], \\ \mu = \frac{(6\rho n-\rho^2 mb)}{m}, \lambda = \frac{(-4m^2\rho^3 b^2+36\rho^2 bn+4m^2\rho^3 ac-108n^2\rho)}{m^2}. \end{cases} \quad (36)$$

For set 6:

$$\begin{cases} v_6(x,y,t) = \frac{-2\rho b}{m} - \frac{4\rho ac}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right]^{-1} \\ \quad -\frac{\rho}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right] \\ u_6(x,y,t) = \frac{-2\mu b}{m} - \frac{4\mu ac}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right]^{-1} \\ \quad -\frac{\mu}{m}\left[\sqrt{\Omega}\tan\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right) - b\right], \\ \mu = \frac{(6\rho n+\rho^2 mb)}{m}, \lambda = \frac{(-4m^2\rho^3b^2-36\rho^2bmn+4m^2\rho^3ac-108n^2\rho)}{m^2}. \end{cases} \tag{37}$$

- **Family 2:** For $\Omega = 4ac - b^2 < 0$, the solutions in Eq. 24 are given by:

$$v(\xi) = -a_{-1}\left(\frac{1}{2c}\left[\sqrt{\Omega}\tan h\left(\frac{1}{2}\sqrt{\Omega}\xi\right)+b\right]\right)^{-1} + a_0 - a_1\frac{1}{2c}\left[\sqrt{\Omega}\tan h\left(\frac{1}{2}\sqrt{\Omega}\xi\right)+b\right],\ u(\xi) = \frac{\mu}{\rho}v. \tag{38}$$

For set 1:

$$\begin{cases} v_7(x,y,t) = \frac{\rho b}{m} - \frac{4\rho ac}{m}\left(\left[\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right)+b\right]\right)^{-1}, \\ u_7(x,y,t) = \frac{\mu b}{m} - \frac{4\mu ac}{m}\left(\left[\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right)+b\right]\right)^{-1}, \\ \mu = \frac{6\rho n}{m},\ \lambda = \frac{\rho(m^2\rho^2b^2-4m^2\rho^2ac-216n^2)}{2m^2}. \end{cases} \tag{39}$$

For set 2:

$$\begin{cases} v_8(x,y,t) = \frac{-\rho b}{m} + \frac{4\rho ac}{m}\left(\left[\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right)+b\right]\right)^{-1}, \\ u_8(x,y,t) = \frac{-\mu b}{m} + \frac{4\mu ac}{m}\left(\left[\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right)+b\right]\right)^{-1}, \\ \mu = \frac{6\rho n}{m},\ \lambda = \frac{\rho(m^2\rho^2b^2-4m^2\rho^2ac-216n^2)}{2m^2}. \end{cases} \tag{40}$$

For set 3:

$$\begin{cases} v_9(x,y,t) = \frac{\rho b}{m} - \frac{\rho}{m}\left(\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right)+b\right), \\ u_9(x,y,t) = \frac{\mu b}{m} - \frac{\mu}{m}\left(\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right)+b\right), \\ \mu = \frac{6\rho n}{m},\ \lambda = \frac{\rho(m^2\rho^2b^2-4m^2\rho^2ac-216n^2)}{2m^2}. \end{cases} \tag{41}$$

For set 4:

$$\begin{cases} v_{10}(x,y,t) = \frac{-\rho b}{m} + \frac{\rho}{m}\left[\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right)+b\right], \\ u_{10}(x,y,t) = \frac{-\mu b}{m} + \frac{\mu}{m}\left[\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^\beta + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^\beta - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^\beta\right)\right)+b\right], \\ \mu = \frac{6\rho n}{m},\ \lambda = \frac{\rho(m^2\rho^2b^2-4m^2\rho^2ac-216n^2)}{2m^2}. \end{cases} \tag{42}$$

For set 5:

From the given sets, the following solutions are derived:

$$\begin{cases} v_{11}(x,y,t) = \frac{2\rho b}{m} - \frac{4\rho ac}{m}\left(\left[\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}\right)\right) + b\right]\right)^{-1} \\ \quad -\frac{\rho}{m}\left[\sqrt{\Omega}\tanh(\frac{1}{2}\sqrt{\Omega}(\frac{\rho}{\beta}((x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)})^{\beta}\right)) + b\right], \\ u_{11}(x,y,t) = \frac{2\mu b}{m} - \frac{4\mu ac}{m}\left(\left[\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}\right)\right) + b\right]\right)^{-1} \\ \quad -\frac{\mu}{m}\left[\sqrt{\Omega}\tanh(\frac{1}{2}\sqrt{\Omega}(\frac{\rho}{\beta}((x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}\left(t+\frac{1}{\Gamma(\beta)})^{\beta}\right)) + b\right], \\ \quad \mu = \frac{(6\rho n - \rho^2 mb)}{m}, \lambda = \frac{(-4m^2\rho^3 b^2 + 36\rho^2 bn + 4m^2\rho^3 ac - 108n^2\rho)}{m^2}. \end{cases} \tag{43}$$

For set 6:

$$\begin{cases} v_{12}(x,y,t) = -\frac{2\rho b}{m} + \frac{4\rho ac}{m}\left(\left[\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}\right)\right) + b\right]\right)^{-1} \\ \quad +\frac{\rho}{m}\left[\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}\right)\right) + b\right], \\ u_{12}(x,y,t) = -\frac{2\mu b}{m} + \frac{4\mu ac}{m}\left(\left[\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}\right)\right) + b\right]\right)^{-1} \\ \quad +\frac{\mu}{m}\left[\sqrt{\Omega}\tanh\left(\frac{1}{2}\sqrt{\Omega}\left(\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}\right)\right) + b\right], \\ \quad \mu = \frac{(6\rho n + \rho^2 mb)}{m}, \lambda = \frac{(-4m^2\rho^3 b^2 - 36\rho^2 bmn + 4m^2\rho^3 ac - 108n^2\rho)}{m^2}. \end{cases} \tag{44}$$

- **Family 3:** For: $\Omega = 4ac - b^2 = 0$, the solutions in Eq. 24 are expressed as:

$$v(\xi) = -a_{-1}(\frac{1}{c}\left[\frac{1}{\xi} + \frac{b}{2}\right])^{-1} + a_0 - a_1\frac{1}{c}\left[\frac{1}{\xi} + \frac{b}{2}\right], u = \frac{\mu}{\rho}v. \tag{45}$$

The following solutions are derived from the given sets:

For set 1:

$$\begin{cases} v_{13}(x,y,t) = \frac{\rho b}{m} - \frac{2\rho ac}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}} + \frac{b}{2}\right]\right)^{-1}, \\ w_{13}(x,y,t) = \frac{\mu b}{m} - \frac{2\mu ac}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}} + \frac{b}{2}\right]\right)^{-1}, \mu = \frac{6\rho n}{m}, \lambda = \frac{\rho(m^2\rho^2 b^2 - 4m^2\rho^2 ac - 216n^2)}{2m^2} \end{cases} \tag{46}$$

For set 2:

$$\begin{cases} v_{14}(x,y,t) = \frac{-\rho b}{m} + \frac{2\rho ac}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}} + \frac{b}{2}\right]\right)^{-1}, \\ u_{14}(x,y,t) = \frac{-\mu b}{m} + \frac{2\mu ac}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}} + \frac{b}{2}\right]\right)^{-1}, \mu = \frac{6\rho n}{m}, \\ \lambda = \frac{\rho(m^2\rho^2 b^2 - 4m^2\rho^2 ac - 216n^2)}{2m^2}. \end{cases} \tag{47}$$

For set 3:

$$\begin{cases} v_{15}(x,y,t) = \frac{\rho b}{m} - \frac{2\rho}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}} + \frac{b}{2}\right]\right), \\ u_{15}(x,y,t) = \frac{\mu b}{m} - \frac{2\mu}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}} + \frac{b}{2}\right]\right), \mu = \frac{6\rho n}{m}, \lambda = \frac{\rho(m^2\rho^2 b^2 - 4m^2\rho^2 ac - 216n^2)}{2m^2}. \end{cases} \tag{48}$$

For set 4:

$$\begin{cases} v_{16}(x,y,t) = -\frac{\rho b}{m} + \frac{2\rho}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}} + \frac{b}{2}\right]\right), \\ u_{16}(x,y,t) = -\frac{\mu b}{m} + \frac{2\mu}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta} + \frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta} - \frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}} + \frac{b}{2}\right]\right), \mu = \frac{6\rho n}{m}, \lambda = \frac{\rho(m^2\rho^2 b^2 - 4m^2\rho^2 ac - 216n^2)}{2m^2}. \end{cases} \tag{49}$$

For set 5:

$$\begin{cases} v_{17}(x,y,t) = \frac{2\rho b}{m} - \frac{2c\rho a}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta}+\frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta}-\frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}}+\frac{b}{2}\right]\right)^{-1} \\ \qquad - \frac{2\rho}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta}+\frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta}-\frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}}+\frac{b}{2}\right]\right), \\ u_{17}(x,y,t) = \frac{2\mu b}{m} - \frac{2c\mu a}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta}+\frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta}-\frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}}+\frac{b}{2}\right]\right)^{-1} \\ \qquad - \frac{2\mu}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta}+\frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta}-\frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}}+\frac{b}{2}\right]\right), \\ \mu = \frac{(6\rho n-\rho^2 mb)}{m}, \lambda = \frac{(-4m^2\rho^3b^2+36\rho^2bn+4m^2\rho^3ac-108n^2\rho)}{m^2}. \end{cases} \tag{50}$$

For set 6:

$$\begin{cases} v_{18}(x,y,t) = \frac{2c\rho a}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta}+\frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta}-\frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}}+\frac{b}{2}\right]\right)^{-1} \\ \qquad - \frac{2\rho b}{m} + \frac{2\rho}{m}\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta}+\frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta}-\frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}}+\frac{b}{2}\right], \\ u_{18}(x,y,t) = \frac{2c\mu a}{m}\left(\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta}+\frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta}-\frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}}+\frac{b}{2}\right]\right)^{-1} \\ \qquad - \frac{2\mu b}{m} + \frac{2\mu}{m}\left[\frac{1}{\frac{\rho}{\beta}(x+\frac{1}{\Gamma(\beta)})^{\beta}+\frac{\mu}{\beta}(y+\frac{1}{\Gamma(\beta)})^{\beta}-\frac{\lambda}{\beta}(t+\frac{1}{\Gamma(\beta)})^{\beta}}+\frac{b}{2}\right], \\ \mu = \frac{(6\rho n+\rho^2 mb)}{m}, \lambda = \frac{(-4m^2\rho^3b^2-36\rho^2bmn+4m^2\rho^3ac-108n^2\rho)}{m^2}. \end{cases} \tag{51}$$

5. Graphical illustrations

This study presents a graphical analysis of the FKD equation to examine how variations in fractional order influence the solutions $v_1(x,y,t)$, $v_5(x,y,t)$, $v_{13}(x,y,t)$ and $v_{17}(x,y,t)$. The analysis is performed using fixed parameter values $c = m = n = \rho = 1$ over the domain $-5 \le x,t \le 5$. Results are visualized in 3D for fractional orders $\beta = 1, \beta = 0.75, \beta = 0.5$, and $\beta = 0.25$ in panels (a-d), while panel (e) presents a corresponding 2D plot at y = 2.

Fig. 1 displays 3D plots of the solutions $v_1(x,y,t)$ in panels (a-d) with parameters $a = 4, b = 2, \mu = 6, \lambda = -114$. Panels (a-d) illustrate the impact of the fractional order of solution $v_1(x,y,t)$. Panels (a) and (b) show strong localized peaks, while panels (c) and (d) display oscillatory behavior and damping effects. Panel (e) presents a 2D slice at $t = 1$, showing the propagation of traveling wave solutions. Similar wave behaviors are observed in the solutions $u_1(x,y,t)$, $u_2(x,y,t)$ and $v_2(x,y,t)$ confirming the consistency of nonlinear dynamics across the system. These visualizations highlight the formation and movement of nonlinear waves, demonstrating the influence of system parameters on wave evolution. Fig. 2 presents a series of visualizations illustrating the behavior of the solution $v_5(x,y,t)$ under specific parameter values $a = 4, b = 2, \mu = 6, \lambda = -114$. Panels (a-d) display 3D representations, while panel (e) provides a 2D plot at t = 1, depicting traveling wave solutions. Panel (a) shows multiple sharp peaks, indicating strong localized singularities and high sensitivity to fractional effects. Panel (b) exhibits pronounced negative spikes, suggesting significant damping. Panel (c) features a single dominant downward spike, representing a highly localized. Panel (d) presents a mix of positive and negative peaks, demonstrating a balance between fractional damping and non-local interactions. The 2D graph shows that when $\beta = 1$, the soliton looks smooth with a few oscillations. However, as β decreases, the oscillatory behavior becomes more noticeable. This pattern is characteristic of nonlinear wave dynamics, where soliton structures evolve based on parameter variations. Additionally, similar wave behavior is observed in the solutions $u_5(x,y,t)$, $v_6(x,y,t)$, and $u_6(x,y,t)$, as demonstrated in their respective 3D visualizations, reinforcing the consistent impact of parameter changes on wave evolution.

Fig. 3 illustrates the shock soliton solution for $v_{13}(x,y,t)$ through 3D and 2D visualizations, highlighting its dynamic behavior under the given parameter settings $a = -4, b = 4, \mu = 6, \lambda = -10$. The physical difference between panels (a) and (b) is minimal, as both exhibit smooth decay. Panel (c) shows a slight reduction in the overall magnitude of the surface values. Panel (d) exhibits a decrease in values, suggesting a stronger damping effect or a greater influence of diffusion. 2D representation in panel (e) further emphasizes that, as the parameter β increases, the magnitude of $v_{13}(x,y,t)$ also increases, indicating a direct proportional relationship.

Fig. 4 provides a detailed visualization of the shock soliton solution for $v_{17}(x,y,t)$ under various conditions. Panels (a-d) offer 3D plots illustrating the evolution of the shock soliton for fixed parameters $a = b = 4, c = 1, \mu = 2, \lambda = -12$. In panel (a), the surface exhibits strong curvature and steep gradients near the boundary, indicating significant variations in the system's behavior.

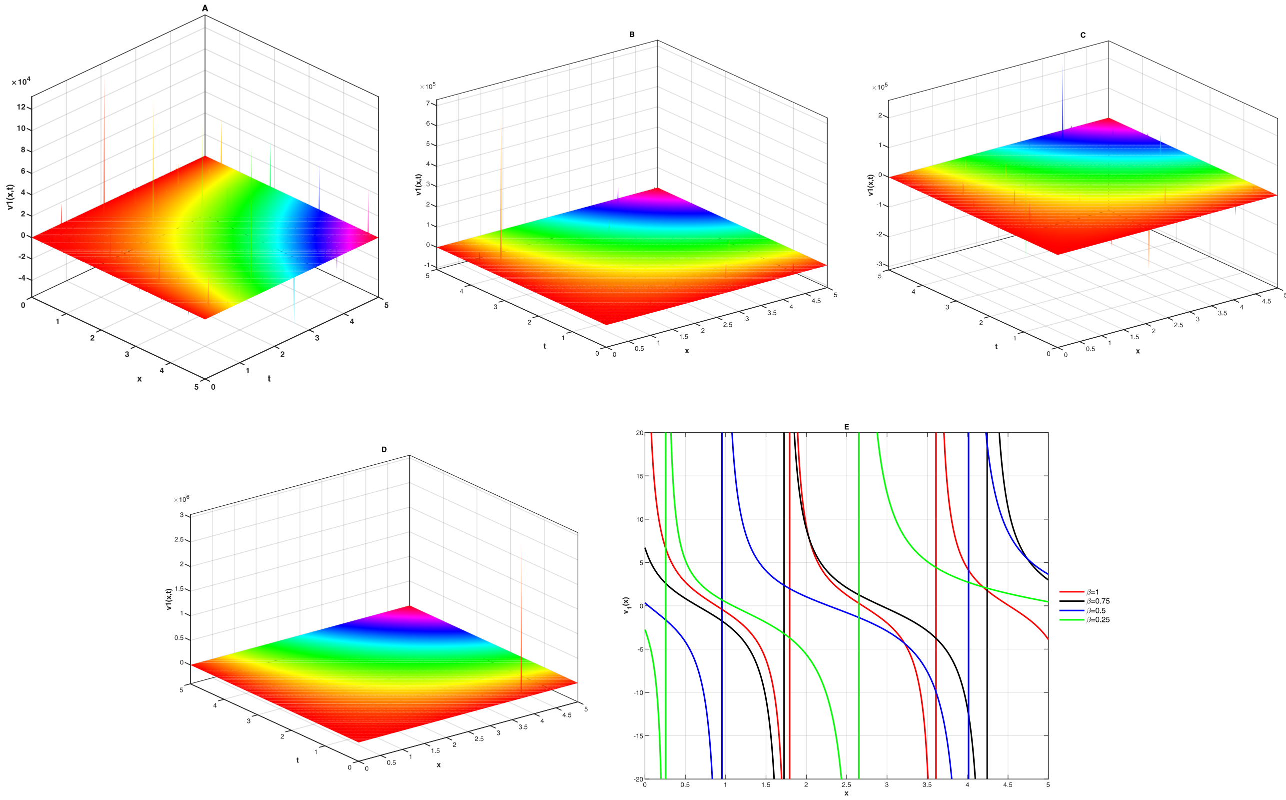

Fig. 1: Panels (A)–(D) illustrate 3D of $v_1(x, y, t)$ corresponding to $\beta = 1, 0.75, 0.5,$ and 0.25 respectively. Panel (E) depicts the 2D of $v_1(x, y, t)$ at a fixed time $t = 1$

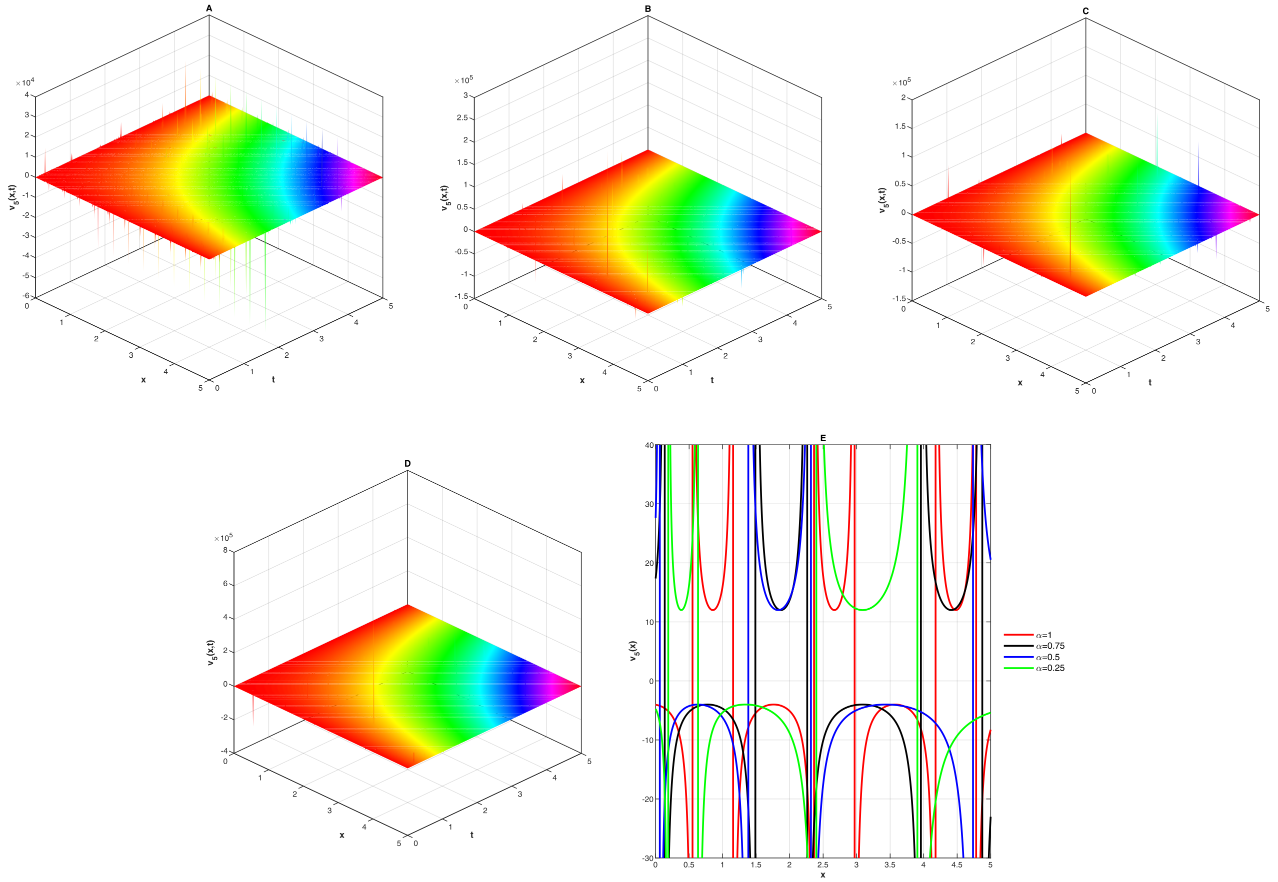

Fig. 2: Panels (A)–(D) illustrate 3D of $v_5(x, y, t)$ corresponding to $\beta = 1, 0.75, 0.5,$ and 0.25 respectively. Panel (E) depicts the 2D of $v_1(x, y, t)$ at a fixed time $t = 1$

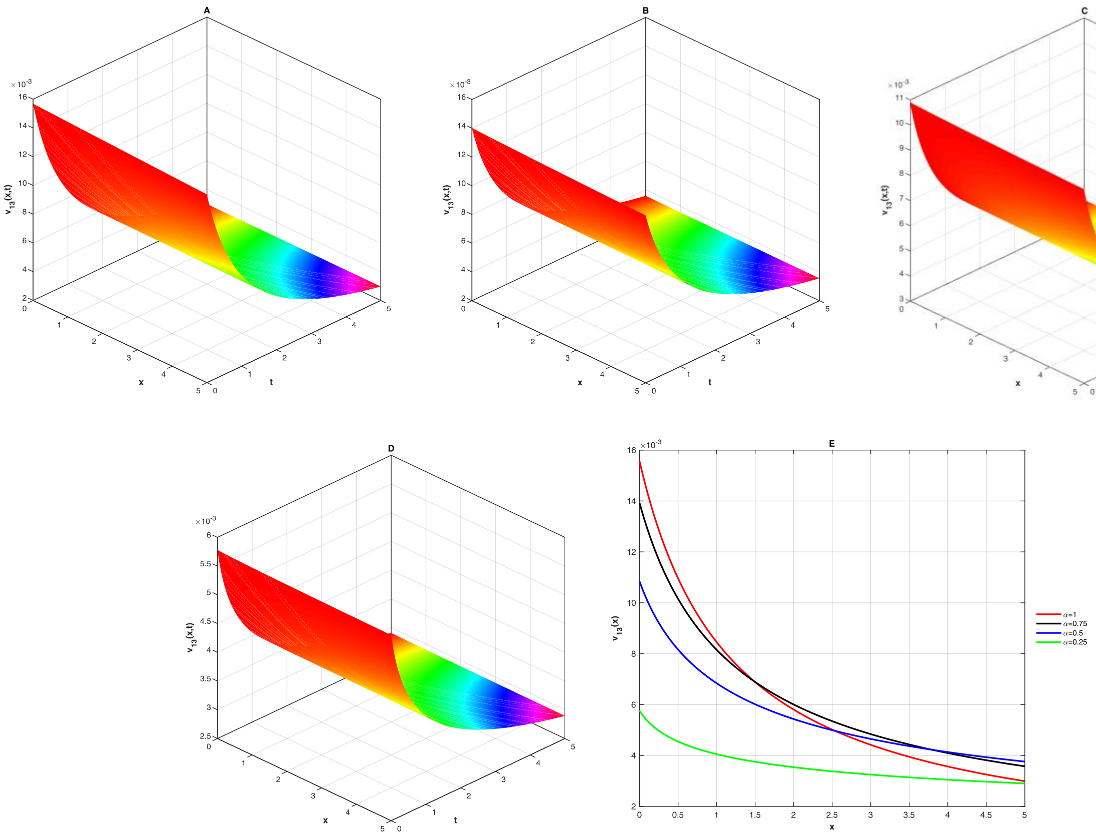

Fig. 3: Panels (A)–(D) illustrate 3D of $v_{13}(x, y, t)$ corresponding to $\beta = 1, 0.75, 0.5$, and 0.25 respectively. Panel (E) depicts the 2D of $v_1(x, y, t)$ at a fixed time $t = 1$

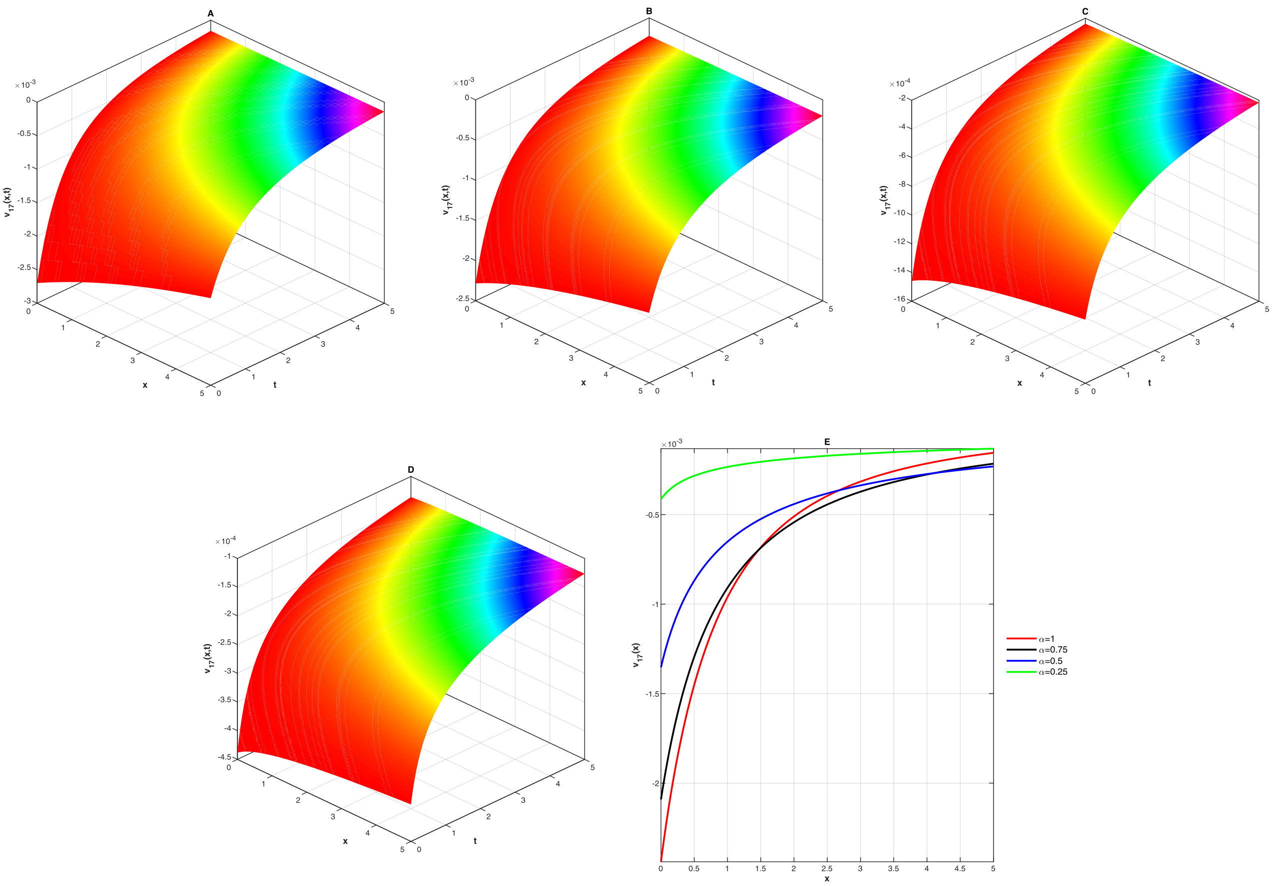

Fig. 4: Panels (A)–(D) illustrate 3D of $v_{17}(x, y, t)$ corresponding to $\beta = 1, 0.75, 0.5$, and 0.25 respectively. Panel (E) depicts the 2D of $v_1(x, y, t)$ at a fixed time $t = 1$

In panel (b), the curvature decreases, suggesting the dissipation of initial disturbances and the transition to a smoother state. The domain expands further in panel (c), and the surface becomes increasingly flatter, implying a reduction in transient effects. In panel (d), the surface appears nearly uniform over a large domain, indicating a stabilized system with minimal variations. Panel (e) presents 2D plots at time $t = 1$ with the same parameters, showing the effect of varying β on the shape of the curve. Specifically, an increase in β is inversely related to the magnitude of $v_{17}(x, y, t)$, indicating that higher values of β reduce the amplitude of the shock soliton. Physically, this suggests that β plays a key role in modulating the intensity of the shock wave or soliton, where higher values of β lead to a more dampened or weaker wave. This could be interpreted in contexts such as fluid dynamics or nonlinear wave phenomena, where the soliton represents a stable, localized wave solution, and β may control energy dissipation or the extent of nonlinearity in the system.

6. Conclusion

In this study, the modified (w/g)-expansion method was successfully employed to derive analytical solutions for the FKD equation, including the beta fractional derivative. Various categories of solutions, including periodic, shock, and traveling wave solutions, were obtained. Graphical representations were provided to visualize and interpret the characteristics of these solutions, offering insights into their behavior. The solutions obtained are consistent with those presented in Wang and Li (2024) when β is set to 1. All calculations in this study were carried out via MAPLE software. Future studies could explore the numerical solution of the FKD equation.

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Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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