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Adaptive Bayesian survival modeling with the Chen-Burr XII distribution: Theory and application to censored COVID-19 data

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A B S T R A C T

This paper introduces an adaptive Type II progressive censoring strategy to improve Bayesian analysis of survival data in life-testing experiments. Using adaptively censored data, the Chen–Burr XII distribution is examined, and Bayesian estimators are derived for its parameters, reliability, hazard rate, and reversed hazard rate under squared error and linear exponential loss functions, assuming independent gamma priors. Credible intervals are constructed to measure parameter uncertainty, and the adaptive Metropolis algorithm is used for Bayesian computation. A simulation study based on four censoring schemes evaluates estimator performance in terms of bias and posterior risk. The results show that estimation efficiency increases with larger sample sizes, more observed failures, and smaller prior variance. Furthermore, the linear exponential loss function with a smaller shape parameter provides more efficient estimates than both larger shape parameters and the squared error loss function. The study also discusses broader methods for developing lifetime distributions, such as transformations and mixtures, and highlights the value of the competing risks approach for modeling events with multiple causes across various fields. The practical usefulness of the proposed methodology is demonstrated through the analysis of real censored lifetime data, including COVID-19 survival data.

1. Introduction

Statistical literature offers diverse methodologies for constructing and generalizing lifetime distributions, including variable transformations, probability integral transforms, and mixture models (Lai, 2013). Among these, the competing risks approach is particularly significant due to its ability to generate models with highly flexible hazard rate functions (hrf), accommodating bathtub and other complex shapes. This approach models scenarios where an event, such as failure, can result from multiple competing causes. Conceptually, these causes compete to trigger the event. Competing risks models are also applicable to series systems, where the lifespan of the system is determined by the minimum lifetime of its independent parts. These models find widespread application in reliability analysis, demography, Biological and medical disciplines, and the field of engineering, and are also referred to as series, additive, or multiple-risk models.

The concept of competing risks has led to the development of various lifetime distributions in the literature. For instance, Xie and Lai (1996) constructed the additive Weibull (AW) distribution through the combination of two Weibull (W) distributions, one with a decreasing and the other with an increasing hrf. Wang (2000) presented the additive Burr XII (ABXII) distribution via the addition of the hrfs of two Burr XII (BXII) distributions. Additionally, Oluyede et al. (2016) introduced the log-logistic W distribution by considering a series system with two components; one has the log-logistic distribution, and another has the W distribution. Mdlongwa et al. (2017) derived the BXII modified W (BXII-MW) distribution. Tarvirdizade and Ahmadpour (2021) introduced the W-Chen (W-C) distribution as a competing risks model between the W and Chen (C) distributions. Osagie and Osemwenkhae (2020) constructed the

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Lomax-W distribution. Kamal and Ismail (2020) presented the flexible W extension-BXII distribution using a combination of the flexible W extension distribution and the BXII distribution in a series system. Thanh Thach and Briš (2021) established the additive C-W distribution, and Makubate et al. (2021) obtained the Lindley-BXII distribution. Abba et al. (2022) presented the flexible additive C-Gompertz distribution from a combination of C and a particular case of Gompertz distributions in a series system. Recently, Méndez-González et al. (2023a) proposed the additive C (AC) distribution, and the additive C-Perks (AC-P) distribution was introduced by Méndez-González et al. (2023b). More recently, Mohammad et al. (2024) constructed the additive xgamma-BXII distribution.

Most recently, Kalantan et al. (2024) introduced a novel competing risks model; the C-BXII distribution, formulated by employing a series system comprising two independently functioning parts. Specifically, the lifetime of the first component, denoted as X_1, has the C distribution, while the lifetime of the second component, X_2, adheres to the BXII distribution. Consequently, the system lifetime, $X = \min\{X_1, X_2\}$, is characterized by the proposed C-BXII distribution. The significance of this distribution is evident in the substantial flexibility and shape diversity exhibited by its hrf and probability density function (pdf). Importantly, the hrf demonstrates crucial forms, including modified bathtub, bathtub, and modified unimodal (decreasing-unimodal) shapes. These characteristics enhance the suitability of the proposed distribution to model lifetime data. Furthermore, the C-BXII distribution encompasses several novel additive models as specific instances, which were not previously known or recognized in statistical literature. Additionally, it includes established models as cases.

The reliability function (rf) and cumulative distribution function (cdf) for the C-BXII distribution are:

$$R(x;\underline{\theta}) = \prod_{i=1}^{2} R_i(x) = \frac{e^{\alpha\left(1-e^{x^\beta}\right)}}{(1+x^c)^k}, \qquad x>0;\ \underline{\theta}>\underline{0}, \qquad (1)$$

and

$$F(x;\underline{\theta}) = 1 - R(x;\underline{\theta}) = 1 - \frac{e^{\alpha\left(1-e^{x^\beta}\right)}}{(1+x^c)^k}, \quad x>0;\ \underline{\theta}>\underline{0}, \qquad (2)$$

where, $\underline{\theta} = (\alpha, \beta, c, k)$ represents a parameter vector.

The hrf for the C-BXII distribution is derived by summing the hrfs of the C and BXII distributions, shown below:

$$h(x;\underline{\theta}) = \alpha\beta x^{\beta-1} e^{x^\beta} + \frac{ckx^{c-1}}{(1+x^c)}, \qquad x>0;\ \underline{\theta}>\underline{0}. \qquad (3)$$

The pdf of the C-BXII distribution is yielded through the connection among pdf, rf and hrf as shown below:

$$f(x;\underline{\theta}) = \left[\alpha\beta x^{\beta-1} e^{x^\beta} + \frac{ckx^{c-1}}{(1+x^c)}\right] \frac{e^{\alpha\left(1-e^{x^\beta}\right)}}{(1+x^c)^k}, \quad x>0;\ \underline{\theta}>\underline{0}. \qquad (4)$$

Also, the reversed hrf (rhrf) of C-BXII distribution is given as:

$$r(x;\underline{\theta}) = \frac{f(x;\underline{\theta})}{F(x;\underline{\theta})} = \frac{\left[\alpha\beta x^{\beta-1} e^{x^\beta} + \frac{ckx^{c-1}}{(1+x^c)}\right] e^{\alpha\left(1-e^{x^\beta}\right)}}{(1+x^c)^k - e^{\alpha\left(1-e^{x^\beta}\right)}}, x>0;\ \underline{\theta}>\underline{0}. \quad (5)$$

Bayesian estimation has gained prominence due to its beneficial characteristics. Notably, it incorporates prior knowledge about unknown parameters, expressed as a joint prior distribution, and combines this with the sample information, represented by the likelihood function (LF), to form the posterior distribution. Furthermore, Bayesian estimation is particularly useful for small sample sizes and censored data. In Bayesian estimation, the squared error (SE) loss function, a symmetric function, and the linear-exponential (LINEX) loss function, an asymmetric function introduced by Varian (1975) and popularized by Zellner (1986), are commonly employed. These loss functions are defined as follows:

$$L_1(\psi, \tilde{\psi}) = \varepsilon(\psi - \tilde{\psi})^2, \qquad (6)$$

and

$$L_2(\psi, \tilde{\psi}) \propto e^{\nu\Delta} - \nu\Delta - 1, \ \ \nu \neq 0, \qquad (7)$$

where, ψ represents the unknown parameter and $\tilde{\psi}$ is its corresponding Bayes estimator, ε is a constant, ν is a constant that defines the loss function's shape and $\Delta = \tilde{\psi} - \psi$.

Life test experiments are commonly conducted in reliability analysis to check the life expectancy of the manufactured product or items/units before products are produced in the market. Due to practical constraints like time, cost, and unexpected factors, it's often impossible to record the failure times of every unit in a life test. This leads to censored samples. In statistical literature, there are various censoring schemes, namely, interval censoring, left censoring, and right censoring. Censoring in life tests occurs in various forms. Left censoring arises when failures occur before a specific time. Interval censoring happens when failure times are known only within a time range. Right censoring, the most prevalent type in reliability analysis, terminates the experiment before all items fail. Right censoring encompasses several subtypes, including Type II and I censoring, progressive censoring, hybrid censoring, adaptive hybrid censoring, and adaptive progressive censoring.

In life-testing and reliability studies, units are often lost or eliminated before failure, either unintentionally or by design. Unintentional losses may stem from accidental breakage, participant withdrawal, or unexpected experiment termination due to factors like budget constraints or facility limitations. Conversely, pre-planned removals free up resources or save time and cost. Ethical considerations may also necessitate removing live units or terminating the experiment. However, Type II, Type I, and hybrid censoring schemes lack the flexibility to remove units during the experiment. To address this, progressive censoring, which allows for removals at various points, has gained popularity. Progressive censoring is categorized into Type II and Type I schemes; for more details (Balakrishnan and Aggarwala, 2000; Balakrishnan, 2007; Balakrishnan and Cramer, 2014).

Like non-progressive censoring, progressive and progressive hybrid censoring schemes have limitations. In Type I progressive and hybrid progressive censoring, the effective sample size, or the number of observed failures, is random and potentially small, invalidating statistical inference. Conversely, Type II progressive and hybrid progressive censoring may result in unpredictable and excessively long experiment durations.

To minimize these concerns, adaptive progressive censoring schemes have been developed. Ng et al. (2009) proposed an adaptive Type II progressive censoring scheme, designed to reduce total test time and failure costs while enhancing statistical analysis efficiency. Lin and Huang (2012) proposed an adaptive Type I progressive censoring scheme.

1.1. Adaptive Type I progressive censoring scheme

Lin and Huang (2012) considered the adaptive Type I progressive censoring scheme. The experiment with the adaptive Type I progressive censoring can be described as follows.

Consider an experiment with n identical units, a predetermined progressive censoring scheme $(R_1, R_2, \cdots, R_m)$, $1 \leq m \leq n$ and a terminating time T. At the first failure, X_1, R_1 surviving units are randomly eliminated. At the second failure, X_2, R_2 are removed, and so on. If the m^{th} failure takes place before T, i.e., $X_m < T$, the experiment continues without further removal until time T, i.e., $R_m = R_{m+1} = \cdots = R_J = 0$, where J represents the number of failures occurring before time T. At time T, all remaining surviving units, $R_J^* = n - \sum_{i=1}^{m-1} R_i - J$, are eliminated, concluding the test. The experimental sequence is shown in Fig. 1. On the other hand, if $T < X_m$, then the experiment is concluded at time T, and the progressive censoring will be $(R_1, R_2, \cdots, R_J)$, where J correspond to the number of failures that occur before time T and all the survival units, $R_J^* = n - \sum_{i=1}^{J} R_i - J$ are removed as displayed in Fig. 2.

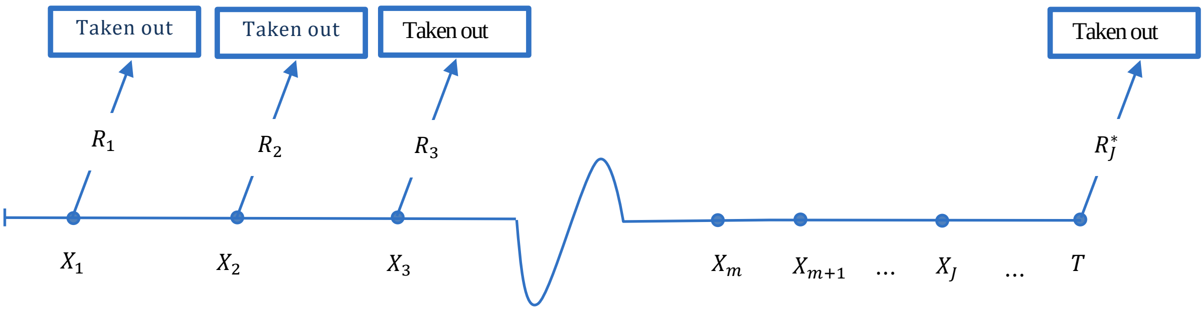

Fig. 1: Experiment relying on the adaptive Type I progressive censoring scheme, $X_m < T$

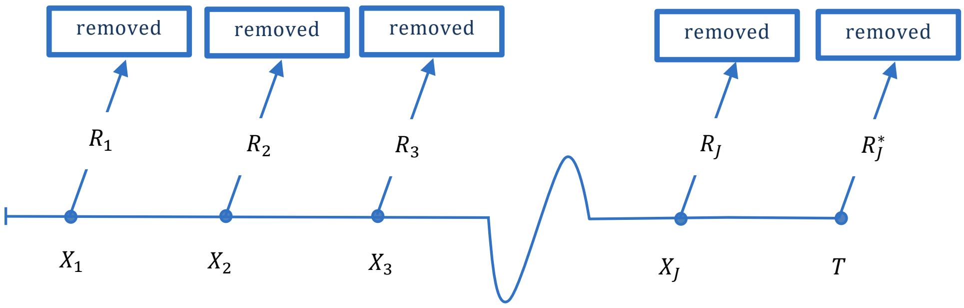

Fig. 2: Experiment using the adaptive Type I progressive censoring scheme, $T < X_m$

1.2. Adaptive Type II progressive censoring

Ng et al. (2009) suggested the adaptive Type II progressive censoring scheme, which combines Type II and Type I progressive censoring. This paper focuses on Bayesian prediction and estimation of the C-BXII distribution using this censoring scheme.

The novelty of this paper lies in several key aspects, which are outlined as follows:

- While the C-BXII distribution has been studied previously, its integration within an adaptive Type II progressive censoring framework under a Bayesian survival modeling scheme is novel. To the best of our knowledge, this is the first attempt to combine the C-BXII distribution with adaptive censoring in a Bayesian context.
- The study further contributes by comparing Bayes estimates under two different loss functions, squared error and linear exponential, highlighting how the shape parameter influences estimator performance. This comparative analysis is rarely addressed in existing literature.
- An adaptive Metropolis algorithm is employed to improve computational efficiency in Bayesian estimation under progressive censoring, offering a

valuable methodological enhancement for practitioners working with complex posterior distributions.

- A comprehensive simulation study, based on four different adaptive censoring schemes, is conducted to evaluate the robustness of the proposed estimators. The analysis of posterior risks and bias under various prior assumptions provides practical guidance for Bayesian analysts.
- Finally, the proposed methodology is applied to real-world censored COVID-19 survival data, demonstrating its practical utility and relevance to current challenges in survival analysis.

The Burr family of distributions, particularly the BXII and its generalizations, has been extensively employed in survival analysis due to their flexibility in modeling various hazard rate shapes (e.g., increasing, decreasing, and bathtub-shaped). Several authors have proposed Bayesian estimation techniques for Burr-type models. However, most of these studies assume fixed or traditional censoring schemes, which may limit their practical effectiveness in modern reliability experiments. Progressive Type II censoring has received attention as a more realistic alternative, yet its full potential under adaptive mechanisms remains underexplored, especially in Bayesian contexts. Moreover, existing works often overlook the impact of different loss functions and prior specifications on estimator behavior.

Recent advancements in Markov Chain Monte Carlo (MCMC) techniques have opened new avenues for Bayesian modeling under complex censoring schemes, but the application of adaptive Metropolis algorithms in this setting remains limited. To the best of our knowledge, no study has yet combined the C-BXII distribution, adaptive Type II progressive censoring, and Bayesian estimation using advanced MCMC approaches. This paper contributes to the literature by filling this gap, offering a comprehensive framework for adaptive Bayesian survival modeling, and supporting it with both simulation and real COVID-19 data.

The paper is structured as follows: Section 2 defines the adaptive Type II progressive censoring scheme, presents an algorithm for generating samples from the C-BXII distribution under this scheme, and reviews relevant literature. Section 3 derives the Bayes estimators for the C-BXII distribution's unknown parameters rf, hrf and rhrf, applying the SE and LINEX loss functions and obtains credible intervals (CIs). Section 4 provides a numerical example to clarify the theoretical results obtained for Bayesian estimation. Moreover, applications are given in Section 5. Finally, a general conclusion is given in Section 6.

1.3. Adaptive Type II progressive censoring scheme

In the adaptive Type II progressive censoring scheme, the predetermined failure count, m, and the progressive censoring scheme $(R_1, R_2, \cdots, R_m)$ are specified. The experimenter sets an optimal total test duration, T, allowing for potential extension. If the m^{th} failure arises before time T, $(X_m < T)$, the experiment terminates at the times of the m^{th} failure, X_m, following standard Type II progressive censoring using the preset $(R_1, R_2, \cdots, R_m)$ scheme, as shown in Fig. 3.

However, if time T passes without reaching m failures, the progressive removal of units is adapted to expedite termination. Specifically, $R_{J+1} = R_{J+2} = \cdots = R_{m-1} = 0$ and $R_m = n - \sum_{i=1}^{J} R_i - m$, where $X_J < T < X_{J+1}$ and and X_J the time of the J^{th} failure, where the J^{th} failure is the last failure before T and $(J+1) < m$. The resulting applied progressive scheme is: $R_1, R_2, \cdots, R_J, 0, \cdots, 0, R_m$. This process is illustrated in Fig. 4. Under this progressive censoring scheme, termination of the experiment occurs early if the $(J+1)^{th}$ failure takes place after time T, while keeping the total experimental duration close to T.

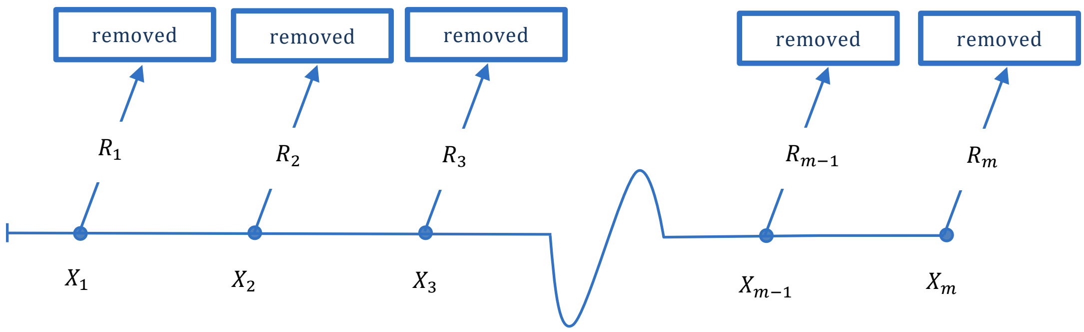

Fig. 3: Experiment designed using a Type II progressive censoring scheme

The key benefit of this censoring method is the acceleration of the experiment when the duration surpasses the pre-set time T, while ensuring enough failures are noted. Thus, it effectively balances the experiment duration, the number of test units, and the efficiency of statistical inference. Additionally, the adaptive Type II progressive censoring scheme serves as a versatile and generalized censoring approach. When $T \to \infty$, the adaptive Type II progressive censoring scheme becomes Type II progressive censoring scheme with progressive censoring $(R_1, R_2, \cdots, R_m)$. While, if $T = 0$, then $R_1 = R_2 = \cdots = R_{m-1} = 0$ and $R_m = n - m$, which leads to Type II censoring scheme. Additionally, this type of progressive censoring scheme can be reduced to the complete case when $T = 0$ and $n = m$.

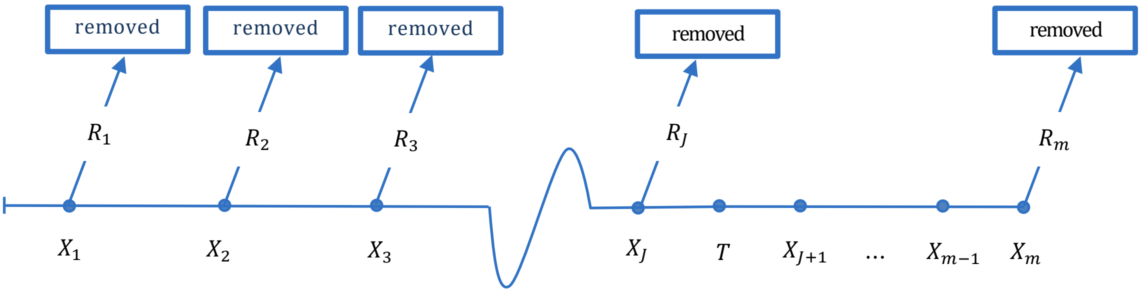

Fig. 4: Experiment relying on the adaptive Type II progressive censoring scheme, $T < X_m$

To obtain an adaptive Type II progressively censored C-BXII sample of size m within a competing risks framework, an algorithm originally presented by Ng et al. (2009) can be modified as given below:

1. Generate a standard Type II progressive censored sample $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ with censoring scheme $(R_1, R_2, \cdots, R_m)$ as follows:
2. Generate m progressively Type II censored order statistics $X_i, i = 1, 2, \cdots, m$ from C-BXII distribution using the algorithm developed by Balakrishnan and Sandhu (1995) as:

- Generate m independent uniform (0,1) observations, $U_{1i}, i = 1, 2, \cdots, m$.
- Set $V_{1i} = U_{1i}^{1/\left(i+\sum_{j=m-i+1}^{m} R_j\right)}$ for $i = 1, 2, \cdots, m$.
- Set $W_{1i} = 1 - \prod_{j=m-i+1}^{m} V_{1i}, i = 1, 2, \cdots, m$. Then, $W_{11}, W_{12}, \cdots, W_{12}$ is a progressive Type II censored sample drawn from a uniform distribution between 0 *and* 1.
- Given the parameters of C-BXII distribution, a progressive Type II censored sample is a solution to the following nonlinear equation. This solution yields the quantile function of C-BXII distribution, which is then used to obtain the censored sample.

$$\alpha\left(1 - e^{x_q^{\beta}}\right) - k\, ln\left(1 + x_q^{c}\right) - ln(1 - q) = 0, \quad 0 < q < 1. \tag{8}$$

- Decide the value of J, where $X_{J:m:n} < T < X_{J+1:m:n}$ and eliminate the sample $X_{J+2:m:n}, X_{J+3:m:n}, \cdots, X_{m:m:n}$.
- Generate the first $m - J - 1$ order statistics from a truncated C-BXII distribution $\frac{f(x)}{1-F(x_{J+1:m:n})}$ with sample size $\left(n - \sum_{i=1}^{J} R_i - J - 1\right)$ as $X_{J+2:m:n}, X_{J+3:m:n}, \cdots, X_{m:m:n}$ by ingenerate $n - \sum_{i=1}^{J} R_i - J - 1$ independent uniform $(0,1)$ observations, $\acute{U}_i, i = 1, 2, \cdots, n - \sum_{i=1}^{J} R_i - J - 1$, then use them to generate a random variable $\acute{X}_i$ from a truncated distribution $\frac{f_1(x)}{1-F_1(x_{J+1:m:n})}$, where $F_1(.)$ and $f_1(.)$ are the cdf and pdf of C-BXII distribution, respectively.

Over the last years, there have been many authors discussing statistical parameter inference of lifetime models based on the adaptive Type II progressive censoring scheme. For instance, Ng et al. (2009) introduced the adaptive Type II progressive censoring scheme and subsequently developed methods for interval and point estimation of the failure rate for exponentially distributed failure times using data obtained through this scheme. Similarly, Lin et al. (2009) explored the maximum likelihood (ML) estimators for the parameters of the W distribution according to the adaptive Type II progressive censoring scheme.

The adaptive progressive Type-II censoring scheme was explored by Cramer and Iliopoulos (2010), who focused on ML estimation for the two- and one-parameter exponential distributions within this framework. Building on this, Hemmati and Khorram (2011) considered a competing risks model with an exponential distribution based on the same censoring scheme. Their analysis involved deriving Bayes and ML estimators for the parameters and constructing the two-sided Bayesian probability intervals. Later, Hemmati and Khorram (2013) applied the adaptive Type II progressive censoring scheme to the log-normal distribution, obtaining the ML estimators for its parameters. Also, Ye et al. (2014) studied general statistical characteristics of this censoring scheme, proposing a bias correction for the ML estimators. They also examined extreme value distributions, derived the Fisher information matrix for the ML estimators using the adaptive Type II progressive censoring scheme, and compared different confidence interval construction methods for the extreme value parameters.

The adaptive Type II progressive censoring scheme was utilized by Sobhi and Soliman (2016) to obtain the Bayesian and ML estimation for the parameters, rf, and hrf of the two-parameter exponentiated W distribution. They further used the asymptotic properties of ML to compute approximate confidence intervals and constructed parametric bootstrap confidence intervals, and derived CIs. Additionally, Nassar and Abo-Kasem (2017) applied the adaptive Type II progressive censoring scheme to present Bayesian and non-Bayesian estimation for the shape and scale parameters of the inverse W distribution, deriving the ML estimators and obtaining Bayes estimators with the SE risk function. Moreover, Ateya and Mohammed (2017) proposed the point and interval estimation for the exponentiated exponential distribution's parameters using an adaptive progressive Type II censoring scheme, considering ML estimation and Bayesian estimation with the SE and LINEX loss functions. They also obtained the approximate confidence intervals and CIs for the parameters.

The adaptive Type II progressive hybrid censoring scheme was employed by Nassar et al. (2018) to obtain the ML and Bayes estimators for the unknown parameters of the W distribution. Assuming independent gamma priorities, they derived the Bayes estimators for both the SE and LINEX loss functions, utilizing Lindley's approximation and MCMC methods for Bayesian computation.

Furthermore, they constructed asymptotic confidence intervals, and two types of parametric bootstrap confidence intervals (within a frequentist framework) compared to the Bayes CIs. Furthermore, Sewailem and Baklizi (2019) investigated the parameter estimation and interval calculation for the log-logistic distribution using the ML and Bayesian approaches with adaptive progressive Type II censored data. The Newton–Raphson method was employed to obtain the ML estimators for the parameters, while the delta method was used to calculate approximate confidence intervals for the rf. For the Bayesian approach, estimators were derived employing the SE loss function, and approximate CIs for both the parameters and the rf were constructed using the MCMC method.

Chen and Gui (2020) considered the estimation problem of the two unknown parameters of C distribution derived from adaptive Type II progressive censoring scheme. They presented the asymptotic confidence intervals and ML estimators. Additionally, they discussed Bayesian point and interval estimation by assuming informative gamma prior distributions of the parameters, under the SE loss function.

Mohammed et al. (2022) introduced the estimation of parameters, rf, and hrf for the Dagum distribution using ML and Bayesian approaches with an adaptive Type II progressive hybrid censoring scheme. Approximate confidence intervals were derived based on the asymptotic properties of the ML estimators. Additionally, the highest posterior density credible intervals were obtained. Alrumayh et al. (2023) estimated the parameters, rf and hrf of the unit-Lindley distribution applying an adaptive Type II progressive censored sample. They used both Bayesian and non-Bayesian estimation techniques. They used ML and maximum spacing procedures as non-Bayesian methods of estimation. Bayes estimators were derived by assuming independent gamma priors and under the SE risk function. Elshahhat et al. (2023) studied the inference problem of the model parameters and the rf and hrf of the Gompertz-Makeham distribution under an adaptive progressively Type II censored samples. They used the ML and Bayesian estimation approaches.

Recently, Dutta et al. (2024) investigated the logistic exponential distribution using adaptively Type II progressively censored data. They developed both Bayes and non-Bayes estimators for the distribution parameters, rf and hrf. For the non-Bayesian approach, they employed the ML and maximum product estimation. Bayesian estimation was performed using SE and LINEX risk functions. Additionally, they constructed asymptotic and highest posterior density confidence intervals for the parameters, rf, and hrf, and provided interval and point Bayesian predictions for future observations. Also, Alotaibi et al. (2024) considered the complementary unit W distribution in the context of the adaptive Type II progressive censoring scheme. They derived point and interval estimators of the model parameters, as well as their main functions, via the ML and Bayesian procedures. The SE loss function and gamma and beta prior distributions were used for obtaining Bayesian estimators.

More recently, Dutta and Kayal (2025) considered the Marshall-Olkin bivariate W distribution as a model of dependent competing risks phenomena. They are concerned with statistical inference of the model parameters based on the adaptive Type II progressive censored samples. The ML estimates of the unknown model parameters were computed by using the Newton-Raphson method. Also, they derived the asymptotic confidence intervals of the parameters by applying the asymptotic normality of the ML estimators. Moreover, Bayes estimates and CIs were calculated under a gamma-Dirichlet prior distribution and under the SE and LINEX loss functions by using MCMC technique.

In addition, Jeon et al. (2025) introduced ML and Bayesian estimators of the parameters and a multicomponent stress–strength reliability model of Pareto distribution under the adaptive Type II progressive censoring scheme. They developed a reference prior with partial information to improve the accuracy of the derived estimators based on the SE risk function. Also, asymptomatic confidence intervals and CIs of the parameters and multicomponent stress-strength reliability were obtained.

2. Bayesian estimation

This section details the derivation of the Bayes estimators for the parameters, rf, hrf, and rhrf of C-BXII distribution. These derivations are performed under the SE and LINEX risk functions, utilizing an adaptive Type II progressively censored sample. Additionally, CIs are obtained.

Suppose that $X_{1:m:n}, X_{2:m:n}, \cdots, X_{m:m:n}$ are the observed ordered sample of size m relying on an adaptive Type II progressive censoring scheme from an n-sized random sample taken from a population with pdf, $f(x;\vartheta)$, and cdf, $F(x;\vartheta)$, where ϑ is the unknown parameter, and the related progressive censoring scheme $R = (R_1, R_2, \cdots, R_m)$. Given $J = j$, the LF is defined by Ng et al. (2009) as:

$$L(\vartheta|\underline{x}) = A \prod_{i=1}^{m} f(x_{i:m:n};\vartheta) \prod_{i=1}^{j} [1 - F(x_{i:m:n};\vartheta)]^{R_i} [1 - F(x_{m:m:n};\vartheta)]^{n-m-\sum_{i=1}^{j} R_i}, \quad (9)$$

where,

$$A = \prod_{i=1}^{m}\left(n - i + 1 - \sum_{k=1}^{\max(i-1,j)} R_k\right) = n(n - R_1 - 1)\cdots(n - R_1 - \cdots - R_j - 1)(n - R_1 - \cdots - R_j - 2)\cdots(n - R_1 - \cdots - R_j - m + 1). \quad (10)$$

Substituting Eqs. 2 and 4 into Eq. 9, the LF of C-BXII distribution resulting from an adaptive Type II progressive censored sample is:

$$L(\underline{\theta}|\underline{x}) = A\left\{\prod_{i=1}^{m}\left[\alpha\beta x_i^{\beta-1}e^{x_i^{\beta}} + \frac{ckx_i^{c-1}}{(1+x_i^c)}\right]\right\}\left[\prod_{i=1}^{m}(1 + x_i^c)^{-k}\right]\prod_{i=1}^{j}\left[(1 + x_i^c)^{-kR_i}\right](1 + x_m^c)^{-k\left(n-m-\sum_{i=1}^{j}R_i\right)} \times \exp\left\{\alpha\left[\sum_{i=1}^{m}\left(1 - e^{x_i^{\beta}}\right) + \sum_{i=1}^{j}\left(1 - e^{x_i^{\beta}}\right)R_i + \left(1 - e^{x_m^{\beta}}\right)\left(n - m - \sum_{i=1}^{j}R_i\right)\right]\right\}, \quad (11)$$

where,

$\underline{\theta} = (\alpha, \beta, c, k), \underline{x} = (x_{1:m:n}, x_{2:m:n}, \cdots, x_{m:m:n}) = (x_1, x_2, \cdots, x_m)$

and A is defined in (10).

If the parameters $\underline{\theta} = (\alpha, \beta, c, k) = (\theta_1, \theta_2, \theta_3, \theta_4)$ of C-BXII distribution are unknown random variables with independent gamma priors with density:

$$\pi_i(\theta_j) \propto \theta_i^{a_i-1}e^{-b_i\theta_i}, \qquad \theta_i > 0;\ a_i, b_i > 0, i = 1, \dots, 4 \quad (12)$$

where, the parameters θ_i have hyperparameters a_i and b_i, $i = 1, \dots, 4$. Then, the joint prior distribution of $\underline{\theta}$ is:

$$\pi(\underline{\theta}) \propto \prod_{i=1}^{4}\pi_i(\psi_i) \propto \prod_{i=1}^{4}\psi_i^{a_i-1}e^{-b_i\psi_i} \propto \alpha^{a_1-1}\beta^{a_2-1}c^{a_3-1}k^{a_4-1}\, e^{-(\alpha b_1+\beta b_2+cb_3+kb_4)},$$
$$\theta_i > 0;\ a_i, b_i > 0, i = 1,2,3,4, \quad (13)$$

where, a_i and b_i, $i = 1,2,3,4$ are the hyperparameters of the joint prior distribution.

Hence, the joint posterior distribution of $\underline{\theta} = (\alpha, \beta, c, k) = (\theta_1, \theta_2, \theta_3, \theta_4)$ can be determined by combining the LF, Eq. 11, and the joint prior, Eq. 13, in the following manner:

$$\pi(\underline{\theta}|\underline{x}) = BL(\underline{\theta}|\underline{x})\pi(\underline{\theta}).$$

So,

$$\pi(\underline{\theta}|\underline{x}) = AB\alpha^{a_1-1}\beta^{a_2-1}c^{a_3-1}k^{a_4-1}\left[\prod_{i=1}^{m}h(x_i;\underline{\theta})\right]\left[\prod_{i=1}^{m}w_i^{-k}\right]\prod_{i=1}^{j}w_i^{-kR_i}w_m^{-k\left(n-m-\sum_{i=1}^{j}R_i\right)} \times \exp\left\{-(\alpha b_1 + \beta b_2 + cb_3 + kb_4) + \alpha\left[\sum_{i=1}^{m}u_i + \sum_{i=1}^{j}u_iR_i + u_m\left(n - m - \sum_{i=1}^{j}R_i\right)\right]\right\}, \quad (14)$$

where, B is the normalizing constant and defined as:

$$B^{-1} = \int_{\underline{\theta}} A\alpha^{a_1-1}\beta^{a_2-1}c^{a_3-1}k^{a_4-1}\left[\prod_{i=1}^{m}h(x_i;\underline{\theta})\right]\left[\prod_{i=1}^{m}w_i^{-k}\right]\prod_{i=1}^{j}w_i^{-kR_i}w_m^{-k\left(n-m-\sum_{i=1}^{j}R_i\right)} \times \exp\left\{-(\alpha b_1 + \beta b_2 + cb_3 + kb_4) + \alpha\left[\sum_{i=1}^{m}u_i + \sum_{i=1}^{j}u_iR_i + u_m\left(n - m - \sum_{i=1}^{j}R_i\right)\right]\right\}d\underline{\theta}, \quad (15)$$

where,

$$\int_{\underline{\theta}} = \int_{\alpha=0}^{\infty}\int_{\beta=0}^{\infty}\int_{c=0}^{\infty}\int_{k=0}^{\infty} \quad and \quad d\underline{\theta} = d\alpha d\beta dc dk. \quad (16)$$

$$u_i = \left(1 - e^{x_i^{\beta}}\right), \quad (17)$$

$$u_m = \left(1 - e^{x_m^{\beta}}\right), \quad (18)$$

$$w_i = (1 + x_i^c), \quad (19)$$

$$w_m = (1 + x_m^c), \quad (20)$$

$h(x_i;\underline{\theta})$ is defined in Eq. 3 and A is defined in Eq. 10. Consequently, the marginal posterior distribution of θ_i is expressed as:

$$\pi_i(\theta_i|\underline{x}) = \int_{\underline{\theta}_l}\pi(\underline{\theta}|\underline{x})d\underline{\theta}_l, \quad l \neq i, \quad l, i = 1, .., 4 \quad (21)$$

2.1. Point estimation

The present subsection addressed point estimation for the parameters rf, hrf, and rhrf of C-BXII distribution. The SE and LINEX loss functions were utilized as an adaptive Type II progressively censored sample.

2.1.1. Bayesian estimation with the squared error loss function

Considering the SE loss function, the Bayes estimator of θ; $\tilde{\theta}_{SE}$, is the estimator that the expected value of the loss function is minimized. This expected loss is termed posterior risk (PR) and is given by

$$PR(\tilde{\theta}_{SE}) = E[L_1(\theta, \tilde{\theta})] = \int_{\theta}L_1(\theta, \tilde{\theta})\pi^*(\theta|\underline{x})d\theta = E(\theta^2|\underline{x}) - [E(\theta|\underline{x})]^2 = V(\theta|\underline{x}). \quad (22)$$

[For more details, see Varian (1975) and Zellner (1986)]. The Bayes estimator of θ which minimizes (22) is given by:

$$\tilde{\theta}_{SE} = E(\theta|\underline{x}). \quad (23)$$

As a result, the Bayes estimators of $\underline{\theta} = (\alpha, \beta, c, k) = (\theta_1, \theta_2, \theta_3, \theta_4)$, when considering the SE loss, are the means of their marginal posterior distributions. Accordingly, utilizing Eq. 21, the Bayes estimators are as follows:

$$\tilde{\theta}_{iSE} = E(\theta_i|\underline{x}) = \int_{\theta_i} \theta_i \pi_i(\theta_i|\underline{x}) d\theta_i = \int_{\underline{\theta}} \theta_i \pi(\underline{\theta}|\underline{x}) d\underline{\theta} = \int_{\underline{\theta}} AB\theta_i \alpha^{a_1-1}\beta^{a_2-1}c^{a_3-1}k^{a_4-1} \left[\prod_{i=1}^{m} h(x_i;\underline{\theta})\right]\left[\prod_{i=1}^{m} w_i^{-k}\right] \prod_{i=1}^{j} w_i^{-kR_i} w_m^{-k\left(n-m-\sum_{i=1}^{j} R_i\right)} \times exp\{-(\alpha b_1 + \beta b_2 + c b_3 + k b_4) + \alpha[\sum_{i=1}^{m} u_i + \sum_{i=1}^{j} u_i R_i + u_m(n-m-\sum_{i=1}^{j} R_i)]\} \quad d\underline{\theta}, \qquad i = 1, \dots, 4, \tag{24}$$

where, $h(x_i;\underline{\theta})$ is given in Eq. 3, A is defined according to Eq. 10, while B represents the normalized constant, the definition of which can be found in Eq. 15, u_i, u_m, w_i and w_m are given, respectively, in Eqs. 17-20 and $\int_{\underline{\theta}}$ and $d\underline{\theta}$ are detailed in Eq. 16. Moreover, the methodology for obtaining the Bayes estimators of rf, hrf, and rhrf involves the utilization of Eqs. 1, 3, 5, and 14, presented in the following steps:

$$\tilde{R}_{SE}(x_0) = E[R(x_0)|\underline{x}] = \int_{\underline{\theta}} R(x_0)\pi(\underline{\theta}|\underline{x}) d\underline{\theta}$$

$$= \int_{\underline{\theta}} AB\alpha^{a_1-1}\beta^{a_2-1}c^{a_3-1}k^{a_4-1} \left[\prod_{i=1}^{m} h(x_i;\underline{\theta})\right]\left[\prod_{i=1}^{m} w_i^{-k}\right] \prod_{i=1}^{j} w_i^{-kR_i} w_m^{-k\left(n-m-\sum_{i=1}^{j} R_i\right)} w_0^{-k} \times exp\{-(\alpha b_1 + \beta b_2 + c b_3 + k b_4) + \alpha[\sum_{i=1}^{m} u_i + \sum_{i=1}^{j} u_i R_i + u_m(n-m-\sum_{i=1}^{j} R_i) + u_0]\} d\underline{\theta}, \tag{25}$$

where,

$$u_0 = \left(1 - e^{x_0^{\beta}}\right) \tag{26}$$

and

$$w_0 = 1 + x_0^c. \tag{27}$$

$$\tilde{h}_{SE}(x_0) = E[h(x_0)|\underline{x}] = \int_{\underline{\theta}} h(x_0)\pi(\underline{\theta}|\underline{x}) d\underline{\theta} = \int_{\underline{\theta}} AB\theta_i \alpha^{a_1-1}\beta^{a_2-1}c^{a_3-1}k^{a_4-1}\, h(x_0;\underline{\theta}) \left[\prod_{i=1}^{m} h(x_i;\underline{\theta})\right]\left[\prod_{i=1}^{m} w_i^{-k}\right] \prod_{i=1}^{j} w_i^{-kR_i} w_m^{-k\left(n-m-\sum_{i=1}^{j} R_i\right)} \times exp\{-(\alpha b_1 + \beta b_2 + c b_3 + k b_4) + \alpha[\sum_{i=1}^{m} u_i + \sum_{i=1}^{j} u_i R_i + u_m(n-m-\sum_{i=1}^{j} R_i)]\} d\underline{\theta}, \tag{28}$$

where, $h(x_0;\underline{\theta})$ is defined in Eq. 3 at time point x_0. and

$$\tilde{r}_{SE}(x_0) = E[r(x_0)|\underline{x}] = \int_{\underline{\psi}} r(x_0)\pi\left(\underline{\psi}|\underline{x}\right) d\underline{\psi} = \int_{\underline{\theta}} AB\theta_i \alpha^{a_1-1}\beta^{a_2-1}c^{a_3-1}k^{a_4-1}\, r(x_0;\underline{\theta}) \left[\prod_{i=1}^{m} h(x_i;\underline{\theta})\right]\left[\prod_{i=1}^{m} w_i^{-k}\right] \prod_{i=1}^{j} w_i^{-kR_i} w_m^{-k\left(n-m-\sum_{i=1}^{j} R_i\right)} \times exp\{-(\alpha b_1 + \beta b_2 + c b_3 + k b_4) + \alpha[\sum_{i=1}^{m} u_i + \sum_{i=1}^{j} u_i R_i + u_m(n-m-\sum_{i=1}^{j} R_i)]\} d\underline{\theta}, \tag{29}$$

where, $r(x_0;\underline{\theta})$ is defined in Eq. 5 at time point x_0.

2.1.2. Bayesian estimation with the linear-exponential loss function

Considering the LINEX loss function, the PR of the Bayes estimator, denoted by $\tilde{\theta}_{LIN}$, for the parameter θ is defined as follows:

$$PR(\tilde{\theta}_{LIN}) = e^{\nu\tilde{\theta}_{LIN}} E(e^{-\nu\theta}|\underline{x}) - \nu(\tilde{\theta}_{LIN} - \tilde{\theta}_{SE}) - 1. \tag{30}$$

For more details about Eq. 30, interested readers are referred to Varian (1975) and Zellner (1986) as the main references.

The Bayes estimator derived via the LINEX loss function, which serves to minimize Eq. 30, is as follows:

$$\tilde{\theta}_{LIN} = \frac{-1}{\nu} ln[E(e^{-\nu\theta}|\underline{x})]. \tag{31}$$

Thus, the Bayes estimators of $\underline{\theta} = (\alpha, \beta, c, k) = (\theta_1, \theta_2, \theta_3, \theta_4)$ is developed using Eq. 14 as follows:

$$\tilde{\theta}_{iLIN} = \frac{-1}{\nu} ln[E(e^{-\nu\theta_i}|\underline{x})], \tag{32}$$

where,

$$E(e^{-\nu\theta_i}|\underline{x}) = \int_{\theta_i} e^{-\nu\theta_i}\pi_i(\theta_i|\underline{x})d\theta_i = \int_{\underline{\theta}} e^{-\nu\theta_i}\pi(\underline{\theta}|\underline{x})d\underline{\theta} =$$
$$AB\alpha^{a_1-1}\beta^{a_2-1}c^{a_3-1}k^{a_4-1}\left[\prod_{i=1}^{m} h(x_i;\underline{\theta})\right]\left[\prod_{i=1}^{m} w_i^{-k}\right]\prod_{i=1}^{j} w_i^{-kR_i} w_m^{-k\left(n-m-\sum_{i=1}^{j}R_i\right)} \times exp\{-\nu\theta_i - (\alpha b_1 + \beta b_2 + cb_3 + kb_4) + \alpha[\sum_{i=1}^{m} u_i + \sum_{i=1}^{j} u_i R_i + u_m(n - m - \sum_{i=1}^{j} R_i)]\}\, d\underline{\theta}, \quad (33)$$

where, $h(x_i;\underline{\theta})$ is given in Eq. 3, A is defined according to Eq. 10, B denotes the normalized constant as found in Eq. 15, and the expressions for u_i, u_m, w_i and w_m are given in Eqs. 17-20, with $\int_{\underline{\theta}}$ and $d\underline{\theta}$ being provided in Eq. 16. In addition, the Bayes estimators of the parameters rf, hrf, and rhrf are determined through applying Eqs. 1, 3, 5, 14, and 32, as detailed subsequently.

$$\tilde{R}_{LIN}(x_0) = \frac{-1}{\nu} ln[E(e^{-\nu R(x_0)}|\underline{x})], \quad (34)$$

where,

$$E(e^{-\nu R(x_0)}|\underline{x}) = \int_{\underline{\theta}} AB\alpha^{a_1-1}\beta^{a_2-1}c^{a_3-1}k^{a_4-1}\left[\prod_{i=1}^{m} h(x_i;\underline{\theta})\right]\left[\prod_{i=1}^{m} w_i^{-k}\right]\prod_{i=1}^{j} w_i^{-kR_i} w_m^{-k\left(n-m-\sum_{i=1}^{j}R_i\right)} \times exp\left\{-(\alpha b_1 + \beta b_2 + cb_3 + kb_4) + \alpha[\sum_{i=1}^{m} u_i + \sum_{i=1}^{j} u_i R_i + u_m(n - m - \sum_{i=1}^{j} R_i)] - \nu\frac{e^{\alpha u_0}}{w_0^k}\right\} d\underline{\theta}, \quad (35)$$

$$\tilde{h}_{LIN}(x_0) = \frac{-1}{\nu} ln[E(e^{-\nu h(x_0)}|\underline{x})], \quad (36)$$

where,

$$E(e^{-\nu h(x_0)}|\underline{x}) = \int_{\underline{\theta}} AB\alpha^{a_1-1}\beta^{a_2-1}c^{a_3-1}k^{a_4-1}\left[\prod_{i=1}^{m} h(x_i;\underline{\theta})\right]\left[\prod_{i=1}^{m} w_i^{-k}\right]\prod_{i=1}^{j} w_i^{-kR_i} w_m^{-k\left(n-m-\sum_{i=1}^{j}R_i\right)} \times exp\{-(\alpha b_1 + \beta b_2 + cb_3 + kb_4) + \alpha[\sum_{i=1}^{m} u_i + \sum_{i=1}^{j} u_i R_i + u_m(n - m - \sum_{i=1}^{j} R_i)] - \nu h(x_0;\underline{\theta})\}\, d\underline{\theta}, \quad (37)$$

and

$$\tilde{r}_{LIN}(x_0) = \frac{-1}{\nu} ln[E(e^{-\nu r(x_0)}|\underline{x})], \quad (38)$$

where,

$$E(e^{-\nu r(x_0)}|\underline{x}) = \int_{\underline{\theta}} AB\alpha^{a_1-1}\beta^{a_2-1}c^{a_3-1}k^{a_4-1}\left[\prod_{i=1}^{m} h(x_i;\underline{\theta})\right]\left[\prod_{i=1}^{m} w_i^{-k}\right]\prod_{i=1}^{j} w_i^{-kR_i} w_m^{-k\left(n-m-\sum_{i=1}^{j}R_i\right)} \times exp\{-(\alpha b_1 + \beta b_2 + cb_3 + kb_4) + \alpha[\sum_{i=1}^{m} u_i + \sum_{i=1}^{j} u_i R_i + u_m(n - m - \sum_{i=1}^{j} R_i)] - \nu r(x_0;\underline{\theta})\}\, d\underline{\theta}, \quad (39)$$

where, $h(x_i;\underline{\theta})$ is given in Eq. 3, A is specified in Eq. 10, B is the normalized constant outlined in Eq. 15, u_i, u_m, w_i and w_m are given, respectively, in Eqs. 17-20, $\int_{\underline{\theta}}$ and $d\underline{\theta}$ are given in Eq. 16, u_0 and w_0 are given in Eqs. 26 and 27 and $h(x_0;\underline{\theta})$ and $r(x_0;\underline{\theta})$ are given in Eqs. 3 and 5 at time point x_0.

2.2. Credible intervals

This subsection is dedicated to the derivation of CIs pertaining to the parameters of C-BXII distribution. As a general principle, a two-tailed, $(L_i(\underline{x}), U_i(\underline{x}))$, $(1-\omega)100\%$ CIs of $\underline{\theta} = (\alpha, \beta, c, k) = (\theta_1, \theta_2, \theta_3, \theta_4)$ are given by:

$$P[L_i(\underline{x}) < \theta_i < U_i(\underline{x})|\underline{x}] = \int_{L_i(\underline{x})}^{U_i(\underline{x})} \pi_i(\theta_i|\underline{x})d\theta_i = 1 - \omega,$$
$$i = 1,2,3,4, \quad (40)$$

where $U_i(\underline{x})$ and $L_i(\underline{x})$ define the upper limit (UL) and the lower limit (LL), respectively, with $(1-\omega)$ representing the credible coefficient.

Given that the marginal posterior distributions for $\underline{\theta} = (\alpha, \beta, c, k) = (\theta_1, \theta_2, \theta_3, \theta_4)$ are specified by Eq. 21, a two-sided CIs with a credible coefficient $(1-\omega)100\%$ credibility level for $\underline{\theta}$ can be given by:

$$P[\theta_i < L_i(\underline{x})|\underline{x}] = \int_{L_i(\underline{x})}^{\infty} \pi_i(\theta_i|\underline{x})d\theta_i = 1 - \frac{\omega}{2}, \quad i = 1,2,3,4, \quad (41)$$

and

$$P[\theta_i < U_i(\underline{x})|\underline{x}] = \int_{U_i(\underline{x})}^{\infty} \pi_i(\theta_i|\underline{x})d\theta_i = \frac{\omega}{2}, \quad i = 1,2,3,4. \quad (42)$$

To derive the two-sided CIs for $\underline{\theta} = (\alpha, \beta, c, k) = (\theta_1, \theta_2, \theta_3, \theta_4)$ with $(1-\omega)100\%$, Eqs. 41 and 42 can be solved numerically.

3. Simulation study

This section is dedicated to the implementation of a statistical simulation aimed at assessing the efficiency characteristics of the derived estimators. The Bayesian calculations within this study will employ the adaptive Metropolis (AM) algorithm, as proposed by Haario et al. (2001).

3.1. The adaptive Metropolis algorithm

Step1: Begin by choosing an initial values vector for the $(l \times 1)$ parameter vector $\underline{\theta}^{(0)}$.
Step 2: For per repetition ξ (from 1 to h, where h is the total number of iterations), generate a candidate parameter vector $\underline{\theta}^*$ from a proposal distribution $j_\xi\left(\underline{\theta}^*|\underline{\theta}^{(0)}, \underline{\theta}^{(1)}, \cdots, \underline{\theta}^{(\xi-1)}\right)$. This algorithm utilizes a Gaussian proposal distribution with mean at the current parameter vector $\underline{\theta}^{(\xi-1)}$ with a covariance matrix $C_\xi = C_\xi\left(\underline{\theta}^{(0)}, \underline{\theta}^{(1)}, \cdots, \underline{\theta}^{(\xi-1)}\right)$.
Step 3: Calculate the acceptance rate (AR) using the formula $AR = \frac{\pi(\underline{\theta}^*|\underline{x})}{\pi(\underline{\theta}^{(\xi-1)}|\underline{x})}$, where, $\pi(\underline{\theta}|\underline{x})$ represents the posterior distribution, disregarding the normalization constant.
Step 4: Accept $\underline{\theta}^*$ as $\underline{\theta}^{(\xi)}$ with probability $min(AR, 1)$. If $\underline{\theta}^*$ is rejected, then $\underline{\theta}^{(\xi)} = \underline{\theta}^{(\xi-1)}$. This can be achieved by sampling a value u from the uniform distribution between 0 and 1 $(U(0,1))$. If $u \leq AR$, then accept the candidate and set the current parameter vector $\underline{\theta}^{(\xi)} = \underline{\theta}^*$. Otherwise, reject the candidate and retain the previous parameter vector, setting $\underline{\theta}^{(\xi)} = \underline{\theta}^{(\xi-1)}$.
Step 5: Perform steps 2 through 4 over a significant number of iterations, h.
Step 6: To mitigate the impact of the starting values, ignore the first M generated parameter vectors $\underline{\theta}$, where M represents the burn-in period.

Employing the AM algorithm, the Bayes estimates and their respective PRs for θ_i, $i = 1,2,3,4$, with the SE and LINEX risk functions can be determined, respectively, as follows:

$$\tilde{\theta}_{iSE} = \frac{1}{h-M}\sum_{j=M+1}^{h} \theta_{ij},$$
$$PR(\tilde{\theta}_{iSE}) = \frac{1}{h-M}\sum_{j=M+1}^{h}\left(\theta_{ij} - \tilde{\theta}_{iSE}\right)^2,$$

and

$$\tilde{\theta}_{iLIN} = \frac{-1}{v} ln\left[\frac{1}{h-M}\sum_{j=M+1}^{h} exp(-v\theta_{ij})\right], \quad PR(\tilde{\theta}_{iLIN}) = e^{v\tilde{\theta}_{iLIN}}\left[\frac{1}{h-M}\sum_{j=M+1}^{h} exp(-v\theta_{ij})\right] - v\left(\tilde{\theta}_{iLIN} - \tilde{\theta}_{iSE}\right) - 1,$$

where, $(\theta_{1j}, \theta_{2j}, \theta_{3j}, \theta_{4j}), j = 1, 2, \cdots, h$, are drawn from the posterior distribution and the initial phase M refers to the burn-in period.

3.1.1. The simulation study is executed by implementing the subsequent steps

- Setting the sample size $n = 50, 60, 100$ and the number of failures $r = (60\%, 80\%)$.
- Three progressive censoring plans are investigated. Scheme I progressively removes $n - m$ items in the first stages, while Scheme III employs the opposite approach, removing, $n - m$ items in the later stages. At Scheme II, $n - m$ items are removed at some middle stages. In brief, the simulation study was carried out considering three censoring schemes:

Scheme I: At the first 10 stages $R's = \frac{m-n}{10}$, else $R's = 0$.
Scheme II: At the middle 10 stages, $R's = \frac{m-n}{10}$, else $R's = 0$.
Scheme III: At the last 10 stages $R's = \frac{m-n}{10}$, otherwise $R's = 0$.
Scheme IV: $R_m = n - m$ and $T = 0$, otherwise $R's = 0$.

Table 1 details the progressive censoring plans utilized for the simulation, where $(\gamma^*3, \delta^*2, \gamma^*4)$ means $(\gamma, \gamma, \gamma, \delta, \delta, \gamma, \gamma, \gamma, \gamma)$. Independent gamma priors were presumed for the Bayesian estimation, and their means and variances are detailed in Table 2. A random sample of size n from C-BXII distribution was generated by using the R function (uniroot) to find the quantile in Eq. 8 of C-BXII distribution. Apply the adaptive Type II progressive censoring methods on the previous step to generate an adaptive Type II censored sample from C-BXII through the generation procedure outlined in Section 2 and for $T = 0.5$. Using $h = 2100$, and $M = 1000$. At each time, $\underline{\theta} = (\alpha, \beta, c, k)$ are drawn from the joint posterior distribution specified in Eq. 14. The AM algorithm was employed to calculate the Bayes estimates of $\underline{\theta}$ with the SE and LINEX loss functions, and their associated PRs were also computed. Using the AM algorithm the 95% CIs are calculated for each parameter and their width.

Table 1: The simulation's progressive censoring methods

n	m	Scheme I	Scheme II	Scheme III	Scheme IV
50	30	(2*10, 0*20)	(0*10, 2*10, 0*10)	(0*20, 2*10)	(0*20, 20*1)
	40	(1*10, 0*30)	(0*15, 1*10, 0*15)	(0*30, 1*10)	(0*30, 10*1)
60	36	(2*8, 4*2, 0*26)	(0*13, 2*8, 4*2, 0*13)	(0*26, 2*8, 4*2)	(0*26, 24*1)
	48	(1*8, 2*2, 0*38)	(0*19, 1*8, 2*2, 0*19)	(0*38, 1*8, 2*2)	(0*38, 12*1)
100	60	(4*10, 0*50)	(0*25, 4*10, 0*25)	(0*50, 4*10)	(0*50, 40*1)
	80	(2*10, 0*70)	(0*35, 2*10, 0*35)	(0*70, 2*10)	(0*70, 20*1)

Table 2: Prior mean and variance of the hyperparameters of the parameters $\underline{\theta} = (\alpha, \beta, c, k)$

$\underline{\theta}$	Prior I		Prior II	
	Mean	Variance	Mean	variance
α	2.6000	(0.0100)	2.6000	(0.0500)
β	0.2000	(0.0100)	0.2000	(0.0500)
c	1.2000	(0.0100)	1.2000	(0.0500)
k	1.5000	(0.0100)	1.5000	(0.0500)

The convergence behavior of a single, arbitrarily chosen generated chain $n = 50, m = 40$, obtained under Scheme I, is shown in Figs. 5–7. Figs. 5–7 comprise trace plots, autocorrelation plots, and histograms illustrating the simulated parameter values and the estimated posterior density for each parameter. Also, the convergence behavior of a single, arbitrarily chosen generated chain $n = 60, m = 48$, obtained under Scheme II, is presented in Figs. 8-10.

Moreover, the convergence behavior of a single, arbitrarily chosen generated chain $n = 60, m = 36$, obtained under Scheme III, is displayed in Figs. 11-13.

Based on the conducted simulation steps, the resulting data are presented in Tables 3-13. Tables 3 - 8 contain the Bayes estimates of $\underline{\theta} = (\alpha, \beta, c, k)$ and the associated PRs, derived via the SE and LINEX risk functions for $(\nu = 0.5, 1.5)$ applying censoring Plans I, II, and III, utilizing Priors I and II. Table 11 exhibits analogous Bayes estimates and posterior risks for $\underline{\theta} = (\alpha, \beta, c, k)$ under Scheme IV (which encompasses Type II censoring and the complete sample as a specific case when $n = m$), also informed by Priors I and II. Furthermore, the 95% credible intervals for the parameters under the first three schemes (I, II, and III) are tabulated in Tables 9, 10, and 12 for both Priors I and II.

Table 3: Bayes estimates and the corresponding PRs with the SE loss function for different sample sizes n, the progressive Schemes I, II, III and based on Prior I

n	m	$\underline{\theta}$	Scheme I		Scheme II		Scheme III	
			Est.	PR	Est.	PR	Est.	PR
50	30	α	2.5846	0.0108	2.5618	0.0113	2.6052	0.0104
		β	1.4143	0.0222	1.1924	0.0179	0.8231	0.0119
		c	1.3073	0.0140	1.2726	0.0106	1.2158	0.0101
		k	1.4733	0.0107	1.4723	0.0109	1.5014	0.0095
	40	α	2.5912	0.0096	2.5886	0.0105	2.5857	0.0094
		β	1.2476	0.0191	1.2558	0.0173	0.9667	0.0128
		c	1.2977	0.0103	1.2886	0.0104	1.2526	0.0095
		k	1.4787	0.0093	1.4699	0.0101	1.4906	0.0091
60	36	α	2.5811	0.0105	2.5743	0.0109	2.5993	0.0100
		β	1.3816	0.0193	1.1504	0.0141	0.9564	0.0112
		c	1.3856	0.0104	1.2751	0.0097	1.2445	0.0089
		k	1.4558	0.0092	1.4873	0.0094	1.5030	0.0095
	48	α	2.5804	0.0101	2.5852	0.0099	2.5876	0.0098
		β	1.4128	0.0187	1.1639	0.0139	1.1613	0.0151
		c	1.3177	0.0089	1.2724	0.0094	1.2779	0.0088
		k	1.4799	0.0090	1.4811	0.0092	1.4847	0.0084
100	60	α	2.5947	0.0093	2.5965	0.0099	2.6170	0.0088
		β	1.4947	0.0155	1.3505	0.0140	1.0334	0.0091
		c	1.3522	0.0100	1.3242	0.0094	1.2960	0.0070
		k	1.4662	0.0090	1.4770	0.0093	1.5482	0.0091
	80	α	2.5966	0.0093	2.5766	0.0097	2.6407	0.0081
		β	1.4801	0.0156	1.2815	0.0135	1.3330	0.0135
		c	1.3467	0.0087	1.3014	0.0084	1.2793	0.0075
		k	1.4488	0.0084	1.5009	0.0091	1.4412	0.0076

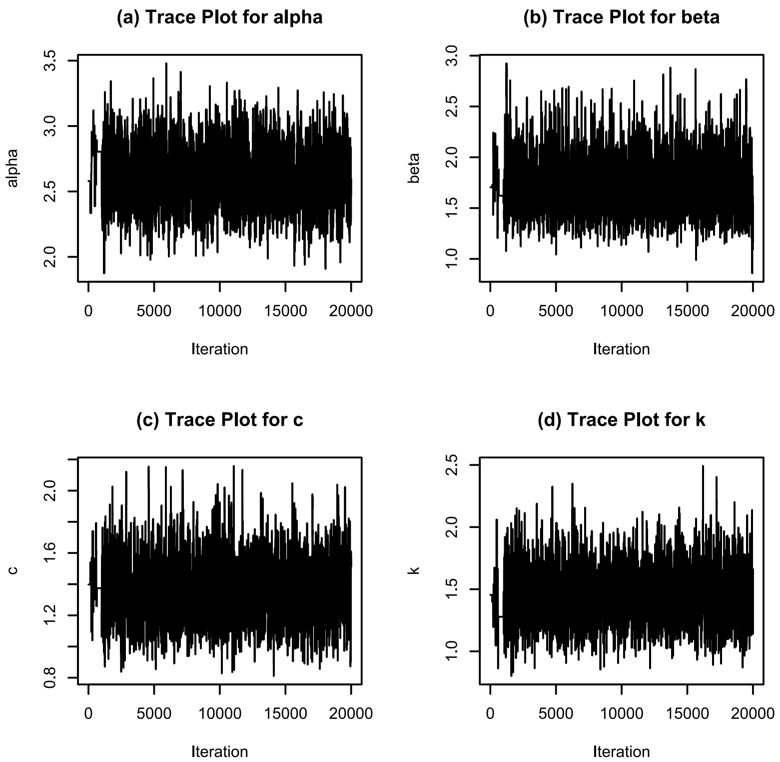

Fig. 5: Trace plots of α, β, c and k under Scheme I

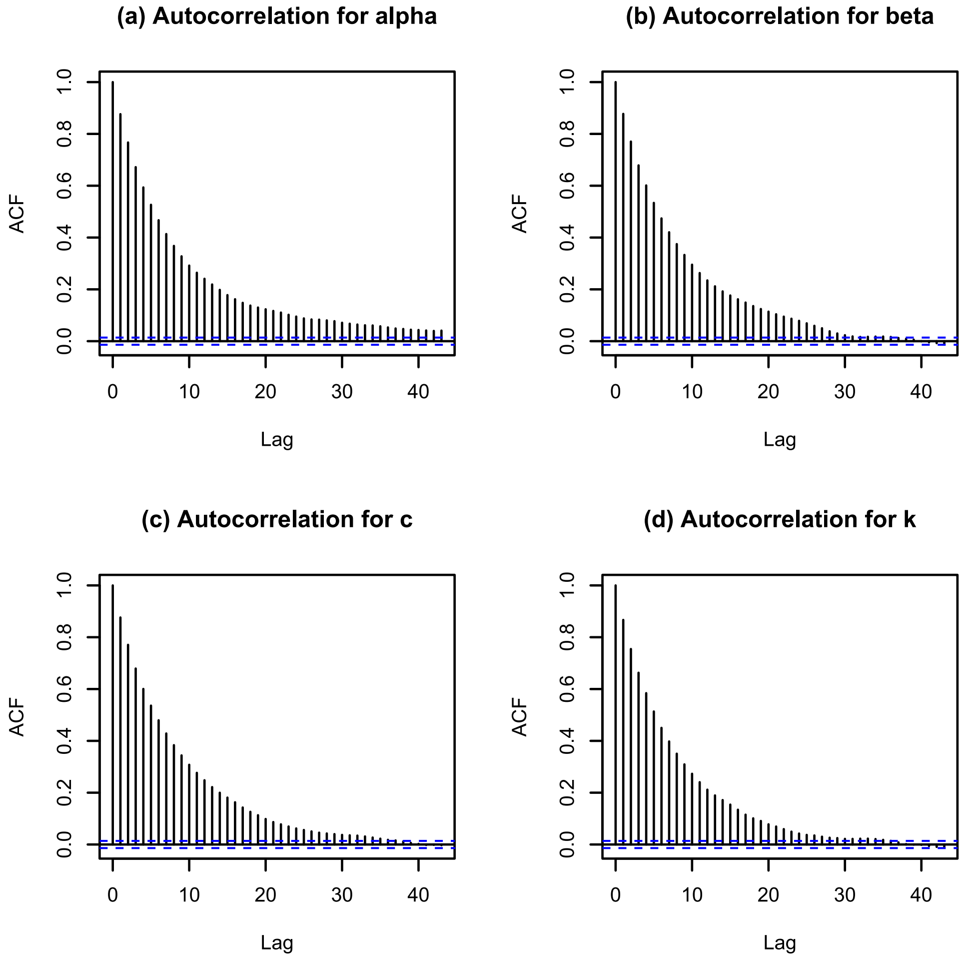

Fig. 6: Autocorrelation plots of α, β, c and k under Scheme I

Table 4: Bayes estimates and the corresponding PRs with the LINEX loss function of $\nu = 0.5$, different sample sizes n, the progressive Schemes I, II, III and based on Prior I

n	m	$\underline{\theta}$	Scheme I		Scheme II		Scheme III	
			Est.	PR	Est.	Est.	PR	Est.
50	30	α	2.5819	0.0027	2.5590	0.0028	2.6027	0.0026
		β	1.4088	0.0056	1.1880	0.0045	0.8202	0.0030
		c	1.3038	0.0035	1.2699	0.0027	1.2132	0.0025
		k	1.4706	0.0027	1.4696	0.0027	1.4990	0.0024
	40	α	2.5888	0.0024	2.5859	0.0026	2.5833	0.0024
		β	1.2429	0.0048	1.2515	0.0044	0.9635	0.0032
		c	1.2952	0.0026	1.2861	0.0026	1.2502	0.0024
		k	1.4763	0.0023	1.4674	0.0025	1.4883	0.0023
60	36	α	2.5785	0.0026	2.5715	0.0027	2.5968	0.0025
		β	1.3768	0.0049	1.1469	0.0036	0.9536	0.0028
		c	1.3268	0.0026	1.2727	0.0024	1.2423	0.0022
		k	1.4535	0.0023	1.4849	0.0024	1.5006	0.0024
	48	α	2.5779	0.0025	2.5828	0.0025	2.5852	0.0025
		β	1.4082	0.0047	1.1605	0.0035	1.1576	0.0038
		c	1.3155	0.0022	1.2700	0.0024	1.2757	0.0022
		k	1.4776	0.0023	1.4789	0.0023	1.4827	0.0021
100	60	α	2.5924	0.0023	2.5941	0.0025	2.6148	0.0022
		β	1.4908	0.0039	1.3470	0.0035	1.0311	0.0023
		c	1.3497	0.0025	1.3219	0.0024	1.2942	0.0018
		k	1.4640	0.0023	1.4747	0.0024	1.5459	0.0023
	80	α	2.5943	0.0023	2.5742	0.0024	2.6386	0.0020
		β	1.4762	0.0039	1.2781	0.0034	1.3296	0.0034
		c	1.3446	0.0022	1.2993	0.0021	1.2775	0.0019
		k	1.4468	0.0021	1.4986	0.0023	1.4393	0.0019

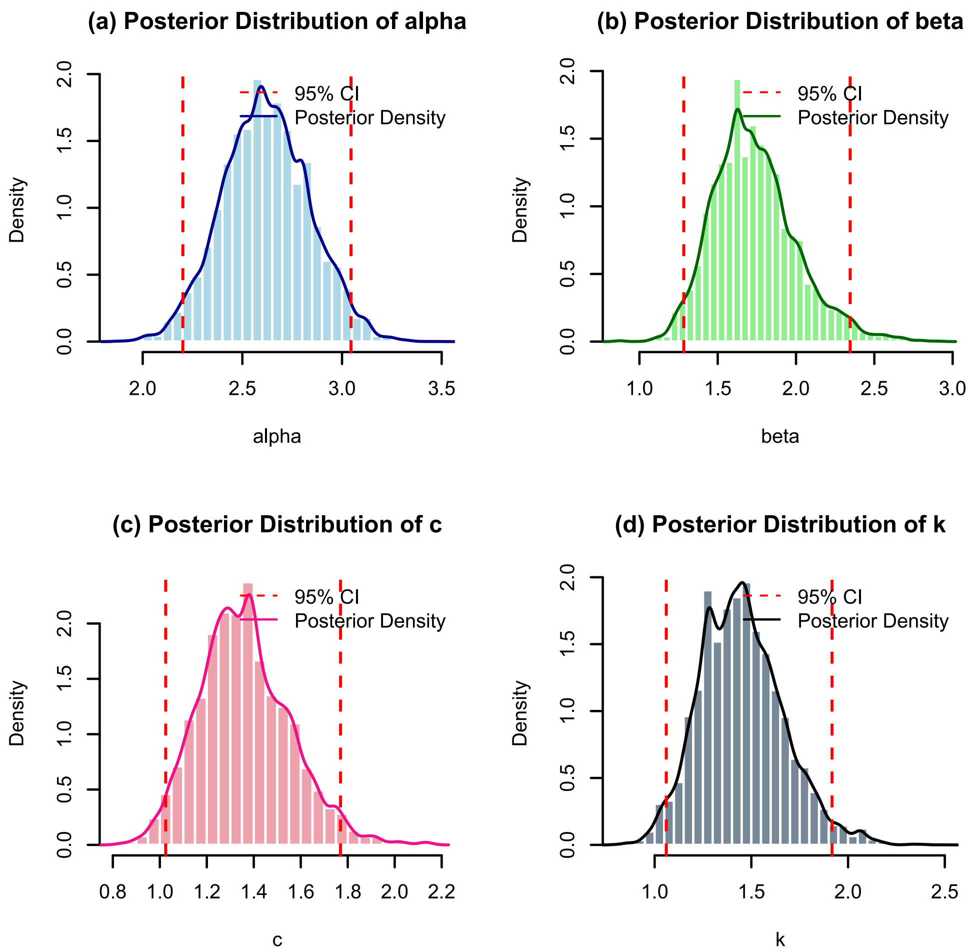

Fig. 7: The histogram, the posterior probability density function and 95% credible intervals for α, β, c and k under Scheme I

Table 5: Bayes estimates and the corresponding PRs with the LINEX loss function of $\nu = 1.5$, different sample sizes n, the progressive Schemes I, II, III and based on Prior I

n	m	θ	Scheme I		Scheme II		Scheme III	
			Est.	PR	Est.	PR	Est.	PR
50	30	α	2.5765	0.0083	2.5533	0.0087	2.5975	0.0080
		β	1.3981	0.0180	1.1793	0.0142	0.8144	0.0094
		c	1.2965	0.0106	1.2647	0.0082	1.2082	0.0078
		k	1.4654	0.0084	1.4642	0.0084	1.4943	0.0073
	40	α	2.5840	0.0073	2.5807	0.0081	2.5786	0.0073
		β	1.2335	0.0151	1.2431	0.0138	0.9572	0.0100
		c	1.2901	0.0079	1.2810	0.0081	1.2454	0.0073
		k	1.4717	0.0072	1.4624	0.0078	1.4838	0.0070
60	36	α	2.5733	0.0080	2.5660	0.0084	2.5918	0.0076
		β	1.3675	0.0155	1.1400	0.0111	0.9481	0.0088
		c	1.3216	0.0080	1.2679	0.0075	1.2379	0.0068
		k	1.4490	0.0071	1.4802	0.0073	1.4959	0.0073
	48	α	2.5729	0.0077	2.5779	0.0077	2.5803	0.0075
		β	1.3991	0.0149	1.1537	0.0109	1.1502	0.0121
		c	1.3111	0.0068	1.2654	0.0073	1.2713	0.0067
		k	1.4731	0.0069	1.4743	0.0071	1.4784	0.0065
100	60	α	2.5878	0.0072	2.5891	0.0076	2.6104	0.0068
		β	1.4832	0.0123	1.3401	0.0109	1.0266	0.0070
		c	1.3447	0.0077	1.3172	0.0072	1.2906	0.0053
		k	1.4596	0.0070	1.4701	0.0072	1.5413	0.0069
	80	α	2.5897	0.0071	2.5695	0.0075	2.6345	0.0062
		β	1.4684	0.0121	1.2716	0.0107	1.3231	0.0106
		c	1.3403	0.0067	1.2951	0.0065	1.2738	0.0058
		k	1.4426	0.0064	1.4961	0.0070	1.4355	0.0059

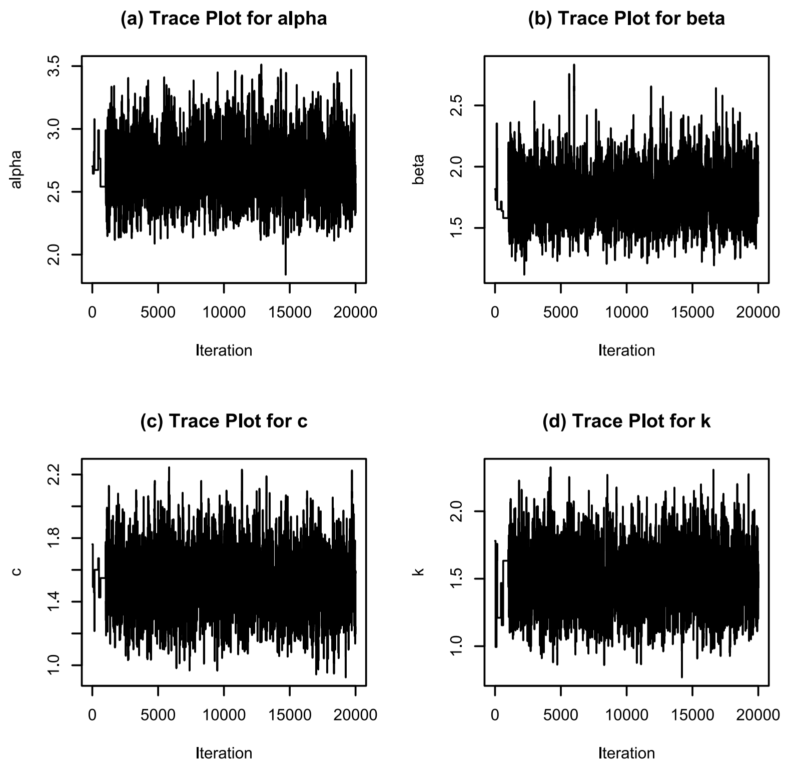

Fig. 8: Trace plots of α, β, c and k with Scheme II

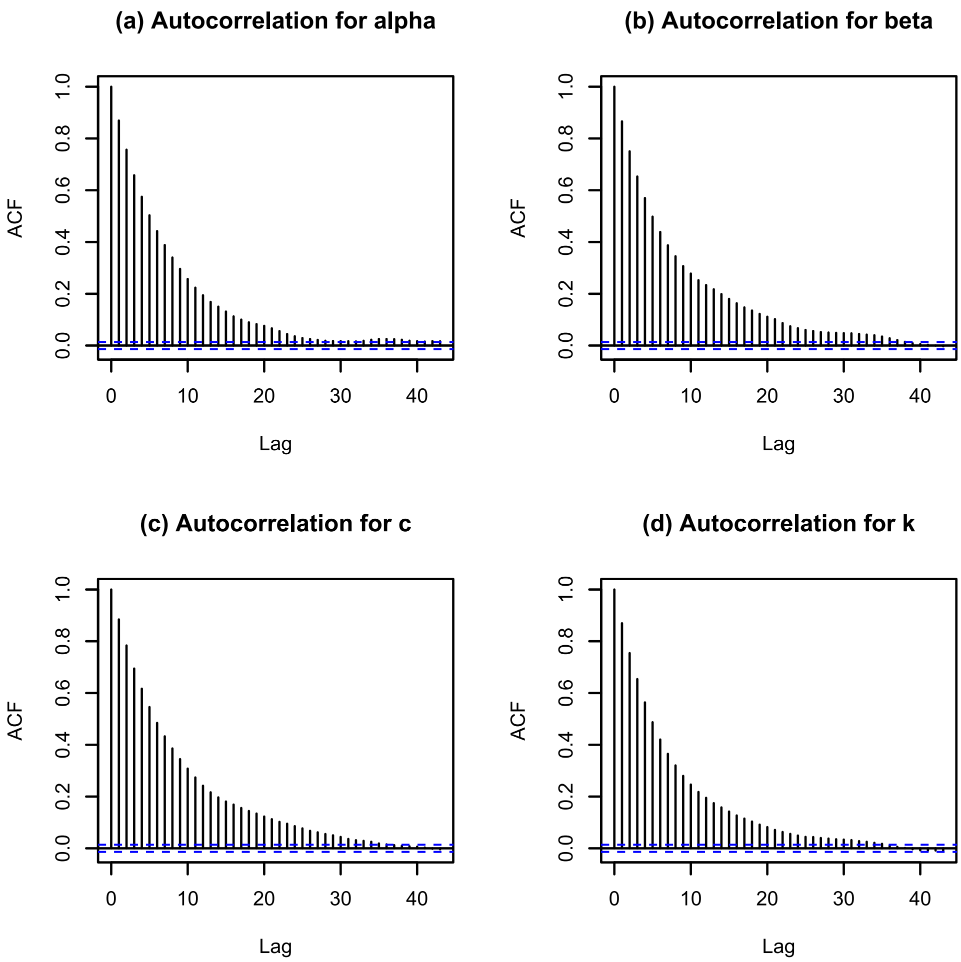

Fig. 9: Autocorrelation plots of α, β, c and k with Scheme II

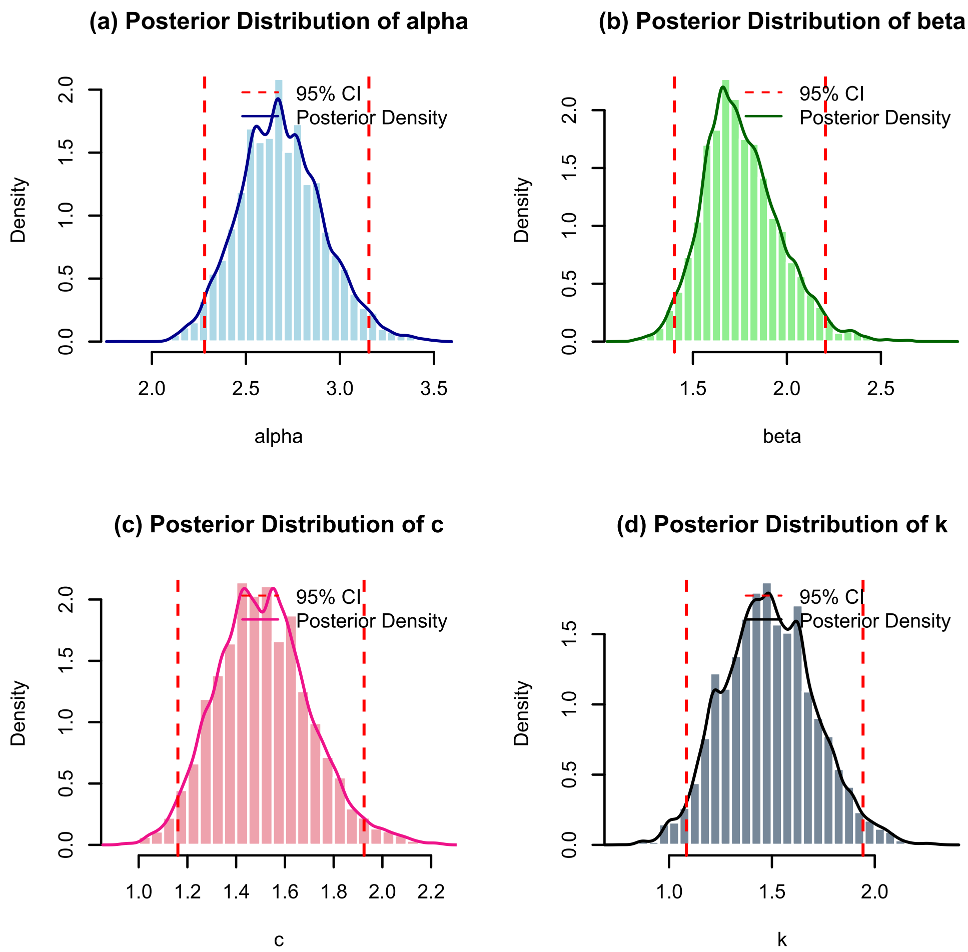

Fig. 10: The histogram, the posterior probability density function and 95% credible intervals for α, β, c and k under Scheme II

Table 6: Bayes estimates, the corresponding PRs with the SE loss function and different sample sizes n, the progressive Schemes I, II, III and based on Prior II

n	m	θ	Scheme I		Scheme II		Scheme III	
			Est.	PR	Est.	PR	Est.	PR
50	30	α	2.6410	0.0513	2.6012	0.0486	2.6647	0.0488
		β	1.8036	0.0608	1.5876	0.0405	1.1548	0.0321
		c	1.4690	0.0459	1.4264	0.0417	1.2975	0.0497
		k	1.4617	0.0469	1.4794	0.0493	1.5151	0.0515
	40	α	2.6529	0.0506	2.6742	0.0480	2.6522	0.0463
		β	1.8290	0.0560	1.4410	0.0324	1.1400	0.0260
		c	1.4250	0.0418	1.4169	0.0379	1.2674	0.0424
		k	1.4394	0.0431	1.4957	0.0463	1.5045	0.0465
60	36	α	2.6387	0.0483	2.6215	0.0478	2.6581	0.0487
		β	1.6963	0.0455	1.3546	0.0304	1.1611	0.0255
		c	1.4778	0.0411	1.3366	0.0403	1.2867	0.0406
		k	1.4719	0.0469	1.5045	0.0470	1.5238	0.0509
	48	α	2.6186	0.0479	2.6495	0.0455	2.7182	0.0480
		β	1.6579	0.0410	1.4578	0.0243	1.3351	0.0255
		c	1.4589	0.0374	1.4441	0.0347	1.3864	0.0341
		k	1.4993	0.0466	1.4569	0.0464	1.5581	0.0497
100	60	α	2.7087	0.0478	2.6865	0.0447	2.7493	0.0505
		β	1.9217	0.0412	1.6107	0.0223	1.0897	0.0150
		c	1.6294	0.0368	1.4620	0.0216	1.3128	0.0399
		k	1.4890	0.0458	1.5026	0.0460	1.5761	0.0497
	80	α	2.6675	0.0463	2.7404	0.0462	2.7671	0.0470
		β	1.8640	0.0401	1.4159	0.0152	1.3748	0.0181
		c	1.5182	0.0338	1.4842	0.0287	1.4287	0.0306
		k	1.4499	0.0453	1.5240	0.0450	1.5883	0.0496

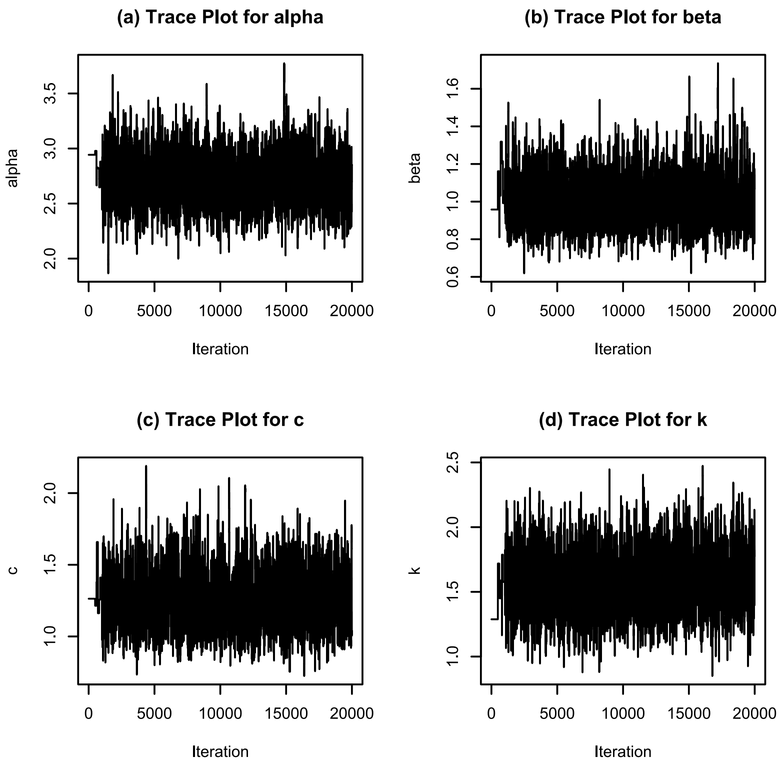

Fig. 11: Trace plots of α, β, c and k under Scheme III

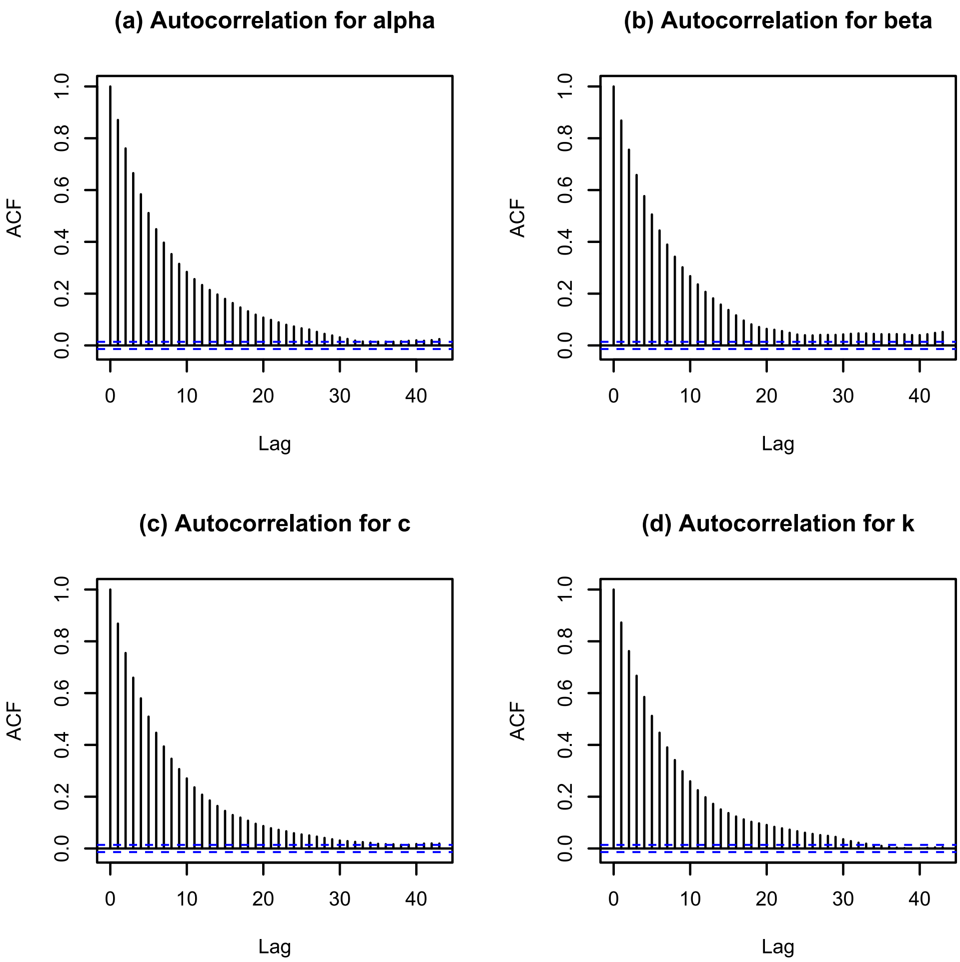

Fig. 12: Autocorrelation plots of α, β, c and k under Scheme III

Table 7: Bayes estimates, the corresponding PRs with the LINEX loss function as $\nu = 0.5$, different sample sizes n, the progressive Schemes I, II, III and based on Prior II

n	m	$\underline{\theta}$	Scheme I		Scheme II		Scheme III	
			Est.	PR	Est.	PR	Est.	PR
50	30	α	2.6283	0.0131	2.5892	0.0124	2.6526	0.0124
		β	1.7888	0.0158	1.5776	0.0104	1.1469	0.0083
		c	1.4577	0.0118	1.4161	0.0107	1.2853	0.0128
		k	1.4501	0.0120	1.4672	0.0127	1.5023	0.0132
	40	α	2.6403	0.0129	2.6623	0.0123	2.6407	0.0118
		β	1.8152	0.0145	1.4331	0.0083	1.1336	0.0067
		c	1.4148	0.0108	1.4075	0.0097	1.2570	0.0109
		k	1.4287	0.0110	1.4842	0.0118	1.4930	0.0119
60	36	α	2.6267	0.0123	2.6097	0.0122	2.6461	0.0124
		β	1.6851	0.0117	1.3472	0.0078	1.1548	0.0065
		c	1.4676	0.0105	1.3266	0.0103	1.2767	0.0104
		k	1.4603	0.0121	1.4929	0.0120	1.5112	0.0130
	48	α	2.6067	0.0122	2.6382	0.0116	2.7063	0.0122
		β	1.6478	0.0105	1.4518	0.0062	1.3288	0.0065
		c	1.4496	0.0095	1.4355	0.0089	1.3780	0.0087
		k	1.4878	0.0119	1.4454	0.0118	1.5458	0.0127
100	60	α	2.6968	0.0122	2.6754	0.0114	2.7368	0.0129
		β	1.9116	0.0106	1.6052	0.0057	1.0860	0.0038
		c	1.6203	0.0094	1.4567	0.0055	1.3030	0.0102
		k	1.4777	0.0117	1.4912	0.0117	1.5637	0.0126
	80	α	2.6560	0.0118	2.7289	0.0118	2.7553	0.0119
		β	1.8542	0.0103	1.4121	0.0038	1.3703	0.0046
		c	1.5099	0.0086	1.4771	0.0073	1.4211	0.0078
		k	1.4387	0.0116	1.5130	0.0115	1.5760	0.0127

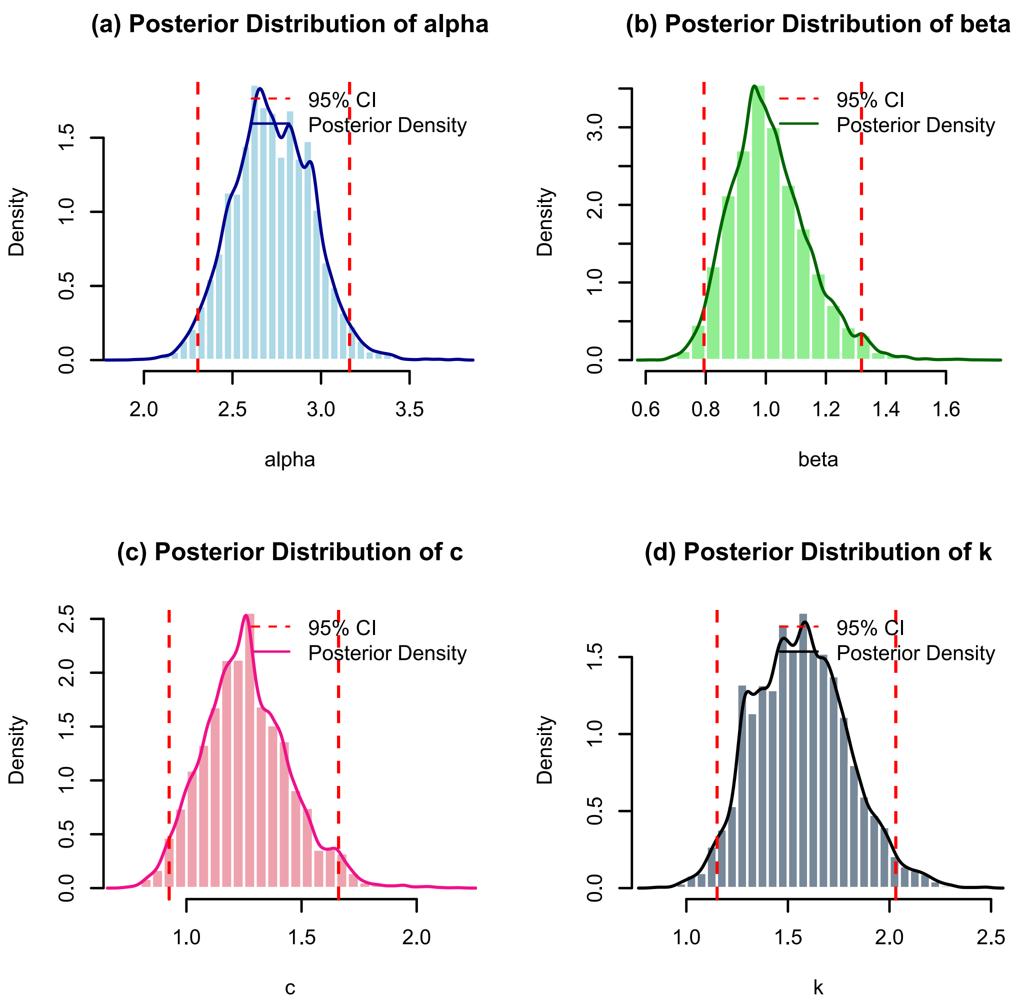

Fig. 13: The histogram, the posterior probability density function and 95% credible intervals for α, β, c and k under Scheme III

Table 8: Bayes estimates, the corresponding PRs with the LINEX loss function as $\nu = 1.5$, different sample sizes n, the progressive Schemes I, II, III and based on Prior II

n	m	θ	Scheme I		Scheme II		Scheme III	
			Est.	PR	Est.	PR	Est.	PR
50	30	α	2.6031	0.0435	2.5657	0.0418	2.6288	0.0416
		β	1.7610	0.0567	1.5584	0.0346	1.1319	0.0274
		c	1.4363	0.0402	1.3962	0.0356	1.2621	0.0436
		k	1.4277	0.0403	1.4439	0.0429	1.4778	0.0448
	40	α	2.6157	0.0432	2.6392	0.0411	2.6181	0.0392
		β	1.7890	0.0499	1.4179	0.0276	1.1215	0.0218
		c	1.3953	0.0366	1.3893	0.0318	1.2369	0.0362
		k	1.4081	0.0367	1.4619	0.0396	1.4709	0.0400
60	36	α	2.6033	0.0411	2.5864	0.0407	2.6226	0.0419
		β	1.6637	0.0397	1.3330	0.0257	1.1426	0.0211
		c	1.4480	0.0350	1.3074	0.0342	1.2575	0.0347
		k	1.4383	0.0410	1.4703	0.0403	1.4869	0.0441
	48	α	2.5834	0.0407	2.6160	0.0383	2.6829	0.0407
		β	1.6285	0.0352	1.4401	0.0198	1.3165	0.0208
		c	1.4315	0.0312	1.4190	0.0292	1.3615	0.0284
		k	1.4650	0.0395	1.4231	0.0396	1.5216	0.0425
100	60	α	2.6734	0.0404	2.6536	0.0376	2.7122	0.0432
		β	1.8926	0.0360	1.5944	0.0180	1.0787	0.0120
		c	1.6028	0.0310	1.4467	0.0180	1.2841	0.0340
		k	1.4559	0.0394	1.4689	0.0390	1.5394	0.0422
	80	α	2.6336	0.0392	2.7064	0.0393	2.7322	0.0396
		β	1.8355	0.0347	1.4046	0.0119	1.3616	0.0147
		c	1.4939	0.0286	1.4631	0.0235	1.4065	0.0254
		k	1.4170	0.0387	1.4916	0.0387	1.5520	0.0424

Table 9: 95% credible intervals along with their width for different sample sizes n,the progressive schemes I, II, III and based on Prior I

n	m	θ	Scheme I			Scheme II			Scheme III		
			UL	LL	Width	UL	LL	Width	UL	LL	Width
50	30	α	2.7869	2.3715	0.4154	2.7775	2.3384	0.4392	2.7979	2.3680	0.4299
		β	1.8122	1.2288	0.5834	1.3613	0.8576	0.5037	1.0990	0.6297	0.4672
		c	1.5046	1.1043	0.4004	1.4673	1.0821	0.3852	1.4458	1.0473	0.3985
		k	1.6927	1.2906	0.4020	1.6852	1.3019	0.3834	1.7392	1.3187	0.4205
	40	α	2.7658	2.3747	0.3911	2.7780	2.3957	0.3823	2.7740	2.3601	0.4139
		β	1.5006	0.9347	0.5658	1.5201	1.0204	0.4997	1.3057	0.8387	0.4500
		c	1.4919	1.0984	0.3935	1.4777	1.0980	0.3797	1.4652	1.0832	0.3820
		k	1.6568	1.2666	0.3903	1.6712	1.2943	0.3769	1.6959	1.2852	0.4107
60	36	α	2.7717	2.3951	0.3766	2.7781	2.3922	0.3859	2.8003	2.4195	0.3808
		β	1.7855	1.2438	0.5418	1.4936	1.0284	0.4652	1.1033	0.7000	0.4033
		c	1.5337	1.1452	0.3885	1.4831	1.1119	0.3711	1.4378	1.0505	0.3872
		k	1.6529	1.2784	0.3745	1.6671	1.2926	0.3745	1.6967	1.3249	0.3717
	48	α	2.7614	2.3857	0.3757	2.7753	2.4047	0.3706	2.8036	2.4252	0.3784
		β	1.6090	1.0714	0.5376	1.3775	0.9570	0.4206	1.2316	0.8343	0.3972
		c	1.5098	1.1292	0.3806	1.4910	1.1505	0.3405	1.4575	1.0868	0.3707
		k	1.6577	1.2875	0.3702	1.6690	1.2937	0.3754	1.6779	1.3086	0.3694
100	60	α	2.7759	2.4068	0.3691	2.7561	2.4126	0.3434	2.7992	2.4341	0.3651
		β	1.9468	1.4282	0.5186	1.4607	1.0219	0.4388	1.2618	0.8730	0.3888
		c	1.5458	1.1707	0.3751	1.4680	1.1169	0.3511	1.4543	1.1125	0.3418
		k	1.6553	1.2857	0.3696	1.6587	1.2994	0.3593	1.7013	1.3701	0.3311
	80	α	2.7687	2.3955	0.3732	2.7894	2.4667	0.3228	2.7606	2.4022	0.3584
		β	1.8387	1.3348	0.5039	1.3862	1.0118	0.3744	1.2831	0.9181	0.3650
		c	1.5497	1.1804	0.3693	1.4722	1.1545	0.3177	1.4464	1.1406	0.3059
		k	1.6392	1.2720	0.3672	1.6829	1.3439	0.3390	1.6426	1.3286	0.3140

Table 10: 95% credible intervals along with their width for different sample sizes n, the progressive schemes I, II and III and based on Prior II

n	m	θ	Scheme I			Scheme II			Scheme III		
			UL	LL	Width	UL	LL	Width	UL	LL	Width
50	30	α	3.0828	2.2261	0.8568	3.0577	2.2115	0.8461	3.0764	2.2154	0.8610
		β	2.2316	1.3707	0.8609	2.0409	1.2000	0.8409	1.4763	0.8251	0.6512
		c	1.9602	1.1683	0.7919	1.8133	1.0247	0.7886	1.7701	0.9084	0.8617
		k	1.9371	1.1017	0.8355	1.9100	1.0613	0.8487	2.0043	1.0950	0.9092
	40	α	3.1113	2.2598	0.8515	3.0822	2.2365	0.8457	3.1026	2.2484	0.8542
		β	2.1374	1.2963	0.8411	1.7699	1.0072	0.7627	1.5479	0.9325	0.6155
		c	1.8982	1.1133	0.7849	1.7375	0.9537	0.7838	1.7820	1.0301	0.7519
		k	1.8959	1.0631	0.8328	1.9541	1.1123	0.8419	2.0098	1.1270	0.8828
60	36	α	3.0674	2.214	0.8534	3.0209	2.1785	0.8425	3.1130	2.2624	0.8506
		β	2.073	1.2216	0.8514	1.9731	1.1949	0.7782	1.5836	0.9440	0.6396
		c	1.9818	1.2007	0.7812	1.9260	1.1474	0.7786	1.7449	0.9907	0.7542
		k	1.9121	1.0864	0.8257	1.9331	1.1015	0.8316	1.9992	1.1180	0.8812
	48	α	3.0299	2.1941	0.8358	3.0504	2.2099	0.8405	3.1185	2.2739	0.8445
		β	2.1759	1.3966	0.7794	2.0562	1.3053	0.7509	1.5472	0.9097	0.6375
		c	1.9637	1.1877	0.7760	1.9191	1.1703	0.7488	1.7938	1.0621	0.7318
		k	1.7991	1.0185	0.7807	1.9354	1.1051	0.8303	1.9940	1.1179	0.8761
100	60	α	3.1567	2.3115	0.8451	3.0592	2.2249	0.8343	3.1887	2.3515	0.8372
		β	2.2237	1.4564	0.7674	1.7881	1.1011	0.6870	1.5777	1.0361	0.5416
		c	1.944	1.1834	0.7606	1.8886	1.1562	0.7324	1.6831	0.9327	0.7504
		k	1.9259	1.0822	0.8437	1.8990	1.0753	0.8237	2.0054	1.1664	0.8390
	80	α	3.0467	2.2495	0.7972	3.1320	2.3068	0.8252	3.1248	2.2932	0.8316
		β	2.0925	1.4272	0.6653	1.8917	1.2590	0.6328	1.5422	1.0338	0.5084
		c	1.9415	1.224	0.7175	1.8747	1.1380	0.7367	1.7757	1.0735	0.7022
		k	1.8673	1.063	0.8043	1.9410	1.1013	0.8397	1.9819	1.1404	0.8415

3.2. Concluding remarks

Analysis of the simulation in Tables 3-13 and Figs. 5-13 reveals that:

- In most cases, increasing the sample size n and the number of observed failures results in lower PRs for the Bayes estimates and narrower CI widths. This reveals that larger and more informative datasets improve both the accuracy and precision of parameter estimation, thereby enhancing the reliability of the Bayesian inference under the proposed adaptive Type II progressive censoring scheme.
- For the same prior means, decreasing the variances of the independent gamma priors results in lower PRs for the Bayes estimates and narrower CI widths. This indicates that more informative priors (i.e., priors with smaller variance) lead to greater estimation accuracy and precision. In this regard, Prior I outperforms Prior II, as its smaller variance produces more reliable and stable parameter estimates.
- The comparison between the SE and LINEX loss functions shows that the PRs obtained under the LINEX loss are consistently lower than those under the SE loss. In terms of minimizing posterior risk, the LINEX loss with $\nu = 0.5$ outperforms the case with $\nu = 1.5$. Moreover, the Bayes estimates obtained under the LINEX loss with $\nu = 0.5$ are very close to those derived using the SE loss function. This confirms that, as the parameter ν approaches zero, the LINEX loss function converges to the SE loss in terms of estimation behavior.
- These results imply that the LINEX loss function, particularly with smaller values of ν, is more effective in reducing posterior risk, offering improved estimation efficiency in asymmetric loss settings. Furthermore, the closeness of LINEX ($\nu = 0.5$) estimates to SE loss estimates suggests that practitioners can achieve both the robustness of asymmetric loss and the familiarity of symmetric loss performance, making LINEX a flexible and advantageous choice in Bayesian survival modeling.
- By comparing the performance of the censoring schemes, Scheme III generally provides the highest efficiency, followed by Scheme II, then Scheme I, with Scheme IV being the least efficient in terms of achieving the smallest PRs and the narrowest CI widths. This implies that Scheme III yields more accurate and precise Bayesian estimates, making it the preferred choice for practical applications where maximizing estimation efficiency is critical.
- The trace plots, autocorrelation plots, histograms, and posterior density plots of the generated parameters (Fig. 5–13) confirm that the MCMC chains have reached convergence and exhibit good mixing. This provides evidence that the posterior samples are stable, independent enough for valid inference, and representative of the true posterior distributions.

4. Application

The model selection criteria have been thoroughly applied in our paper to justify the performance of the proposed model. In particular, following the approach of Kalantan et al. (2024), the model's performance was assessed using several widely recognized metrics, including the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Corrected Akaike Information Criterion (CAIC), in addition to the -2 log-likelihood (-2L) statistic and goodness-of-fit measures such as the Kolmogorov-Smirnov (K-S), Anderson-Darling (A-D), and Cramér-von Mises (C-V) statistics with their corresponding p-values.

These criteria were used to compare the proposed C-BXII distribution with several known distributions, namely: AC-P, W-C, BXII-MW, AW, AC, ABXII, C, and BXII distributions. The findings demonstrate the superiority of the C-BXII distribution, which consistently yielded the lowest values for AIC, BIC, and CAIC, as well as the highest p-values, thereby confirming its strong performance in modeling the considered datasets.

This section assesses the performance and practical utility of the proposed Bayesian estimation approach through its application to a real-world dataset concerning the survival durations of patients diagnosed with Coronavirus disease 2019 (COVID-19) in China. The dataset, originally presented by Liu et al. (2021) and later employed by Kalantan et al. (2024) for illustrating theoretical aspects of Bayesian inference, records the survival times (in days) of 53 critically ill patients hospitalized between January and February 2020. These patients were admitted to intensive care units during the early phase of the outbreak, when treatment protocols were still evolving, and mortality risk was high, making the dataset particularly relevant for survival modeling under censoring.

The data are: 0.054, 0.064, 0.704, 0.816, 0.235, 0.976, 0.865, 0.364, 0.479, 0.568, 0.352, 0.978, 0.787, 0.976, 0.087, 0.548, 0.796, 0.458, 0.087, 0.437, 0.421, 1.978, 1.756, 2.089, 2.643, 2.869, 3.867, 3.890, 3.543, 3.079, 3.646, 3.348, 4.093, 4.092, 4.190, 4.237, 5.028, 5.083, 6.174, 6.743, 7.274, 7.058, 8.273, 9.324, 10.827, 11.282, 13.324, 14.278, 15.287, 16.978, 17.209, 19.092 and 20.083.

The recorded times range from a few hours (0.054 days) to approximately 20 days, reflecting substantial heterogeneity in patient outcomes and suggesting the need for a flexible survival model capable of capturing both short-term and extended survival behavior. In the context of epidemiology, such heterogeneity can indicate variations in disease progression rates, accompanying diseases, and treatment responses. The C–BXII distribution's ability to accommodate multiple hazard rate shapes, monotonic and non-monotonic, makes it well-suited for modeling this complexity.

Using the adaptive Type II progressive censoring Bayesian approach, the model parameters are estimated under both the SE and LINEX loss functions, with results presented alongside 95% CI. The adaptive Metropolis algorithm facilitated efficient posterior sampling, ensuring convergence and stable estimation. The posterior estimates provide insight into the underlying hazard dynamics: the estimated hazard function initially increases during the early days of hospitalization, peaks within the first week, and then declines for patients who survive longer, aligning with clinical observations of COVID-19 progression in critical cases. The corresponding rf estimates survival probabilities at different time points, which can be directly interpreted for medical decision-making.

From a practical perspective, these results demonstrate that the proposed framework can support real-time predictive modeling in epidemic settings. In life-testing terms, the adaptive censoring strategy improves efficiency by allowing earlier termination of data collection without sacrificing estimation quality. In epidemiological contexts, the method provides accurate survival estimates under incomplete data conditions, offering critical insights for resource allocation, patient prioritization, and evaluating the effectiveness of control strategies.

The suitability and appropriateness of modeling the C-BXII distribution to the present dataset was conducted using the goodness of fit statistical tests: the K-S, A-D, C-V. The calculated statistics and their associated p-values, AIC, BIC, and CAIC are presented in Table 14. Based on these findings, it can be inferred that the C-BXII distribution provides an appropriate model for this dataset. Adaptive Type II progressively censored data was generated with a total $n = 53$ initial units and observed, $m = 32$ failures, according to censoring scheme III and the progressive censoring arrangement $(0^{*}22, 2^{*}9, 3^{*}1)$. Therefore, the survival times of the generated adaptive Type II progressive censored sample for $m = 32$ is: 0.054, 0.064, 0.087, 0.087, 0.235, 0.352, 0.364, 0.421, 0.437, 0.458, 0.479, 0.548, 0.568, 0.704, 0.787, 0.796, 0.816, 0.865, 0.976, 0.976, 0.978, 1.756, 1.978, 2.089, 2.643, 2.869, 3.079, 3.348, 3.543, 3.646, 3.867 and 3.890. The suitability of the C-BXII distribution for the adaptive Type II progressively censored sample is assessed using goodness-of-fit statistical tests (K-S, A-D, and C-V) along with their associated p-values. The corresponding AIC, BIC, and CAIC values are presented in Table 15.

From Table 15, it is evident that the C-BXII distribution provides an excellent fit to the adaptive Type II progressively censored data. Fig. 14 presents the empirical and theoretical densities and cdfs for the C-BXII distribution, along with the corresponding Q–Q and P–P plots. Fig. 15 illustrates the empirical rf obtained via the Kaplan–Meier method alongside the fitted rf. These results confirm that the C-BXII distribution offers a suitable fit for the generated adaptive Type II progressively censored sample from China.

Based on the generated adaptive Type II progressively censored sample, the parameters' Bayes estimates are determined via both the SE and LINEX risk functions, in which ν takes the values 0.5 and 1.5. Also, 95% credible intervals and their widths are obtained by assuming an independent informative gamma prior with means $(0.5, 0.25, 1.5, 0.5)$ and equal variances $(0.01, 0.01, 0.01, 0.01)$. All these results are presented in Table 16.

Figs. 16-18 display the trace plots, autocorrelation plots, and histograms of the generated parameters and their posterior densities per parameter for the application, indicating convergence.

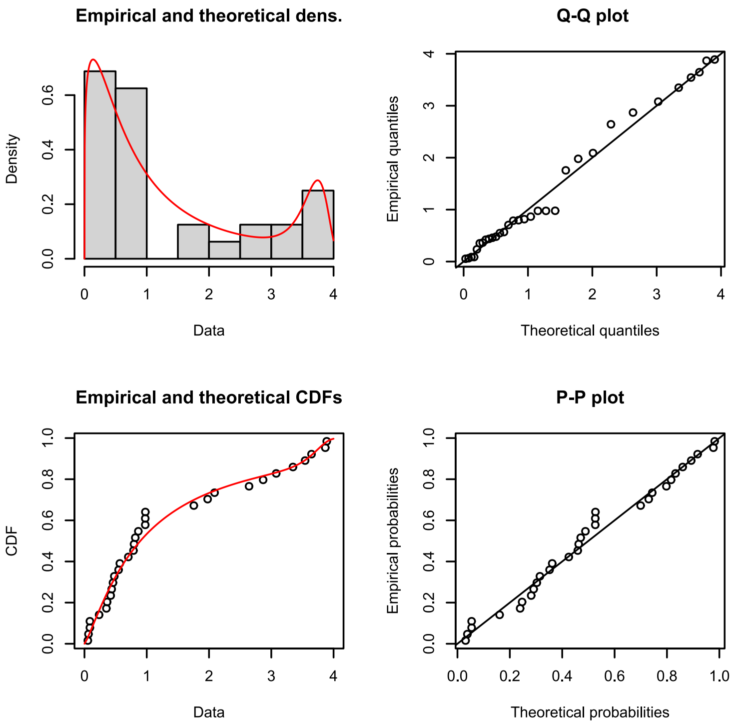

Fig. 14: Empirical and theoretical density and cdf plots, Q – Q and P – P plots of C-BXII distribution for China adaptive Type II progressively censored data

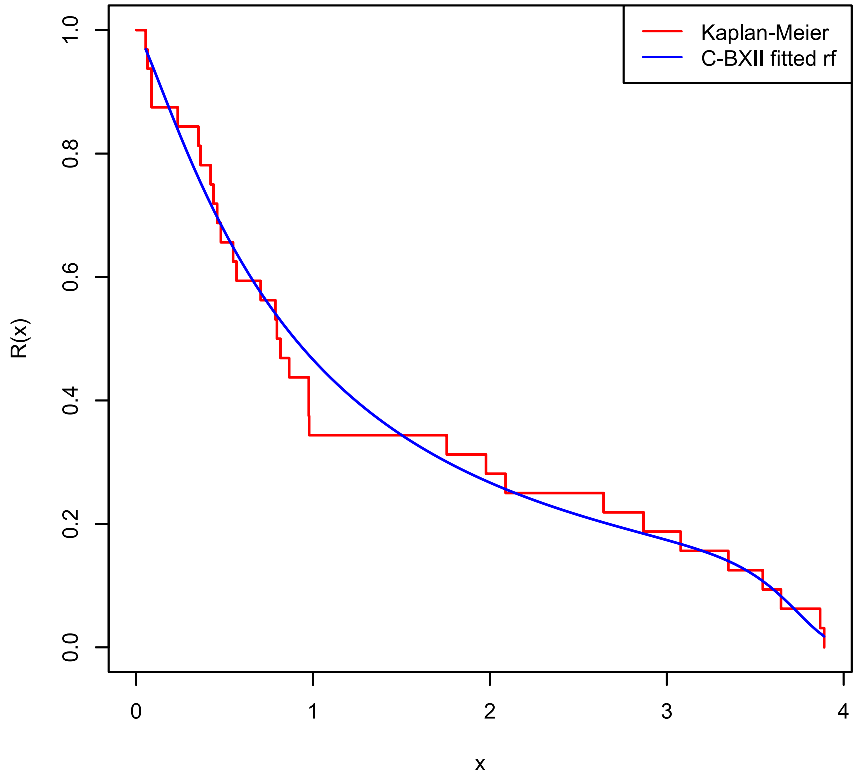

Fig. 15: Kaplan Meier estimate and the fitted rf of C-BXII distribution for China adaptive Type II progressively censored data

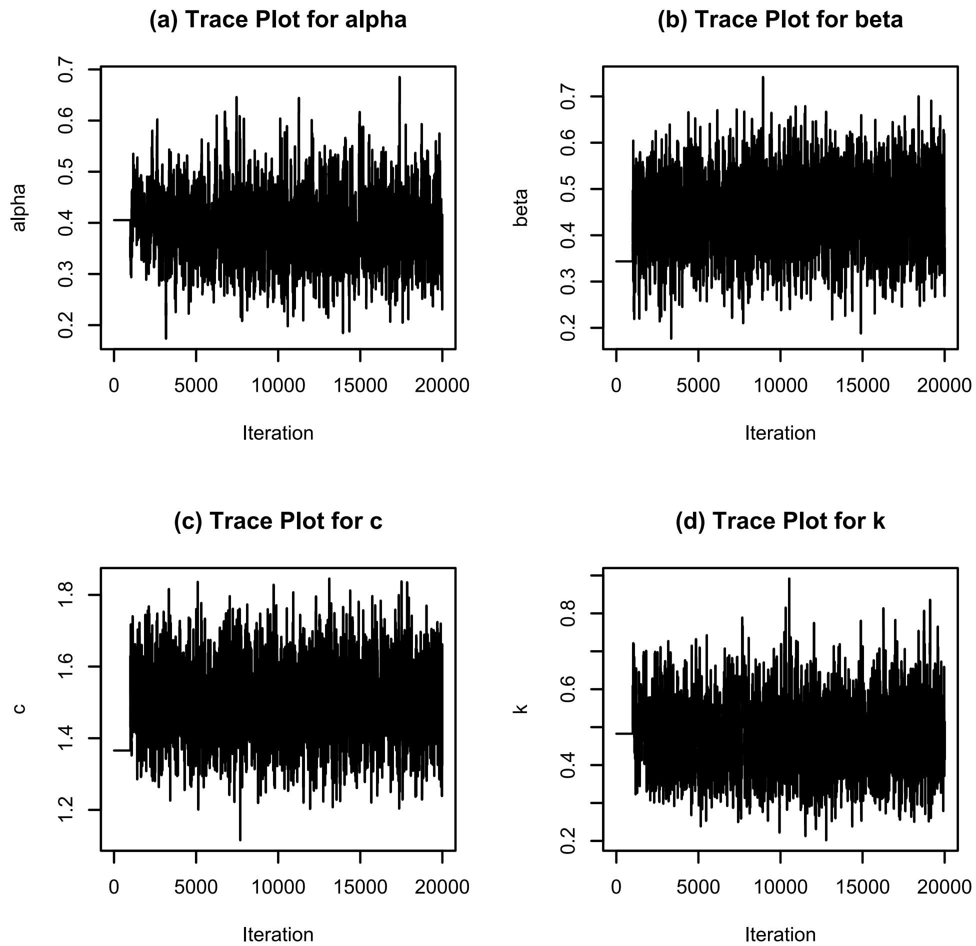

Fig. 16: Trace plots of α, β, c and k for the application

Table 11: Bayes estimates and the corresponding PRs via the SE loss function with different sample sizes n, under Scheme IV and based on Priors I and II

n	m	$\underline{\theta}$	Prior I		Prior II	
			Est.	PR	Est.	PR
50	30	α	2.6091	0.0101	2.6755	0.0512
		β	0.9240	0.0121	1.1725	0.0291
		c	1.2517	0.0092	1.3073	0.0367
		k	1.5112	0.0106	1.5487	0.0526
	40	α	2.5916	0.0096	2.6676	0.0497
		β	0.9672	0.0112	1.1488	0.0245
		c	1.2509	0.0087	1.3151	0.0384
		k	1.4927	0.0098	1.5372	0.0472
	50	α	2.5507	0.0087	2.5716	0.0401
		β	1.0793	0.0166	1.1658	0.0230
		c	1.2490	0.0082	1.2739	0.0380
		k	1.4761	0.0084	1.4772	0.0470
60	36	α	2.5933	0.0135	2.7089	0.0483
		β	0.9051	0.0108	0.9609	0.0151
		c	1.2378	0.0087	1.2569	0.0326
		k	1.5041	0.0092	1.5839	0.0495
	48	α	2.6043	0.0093	2.6903	0.0468
		β	1.0323	0.0145	1.1017	0.0188
		c	1.2593	0.0081	1.2645	0.0365
		k	1.5023	0.0095	1.5507	0.0485
	60	α	2.4911	0.0040	2.5557	0.0441
		β	1.6389	0.0228	1.5574	0.0387
		c	1.3388	0.0039	1.4022	0.0371
		k	1.4584	0.0070	1.4510	0.0403
100	60	α	2.6191	0.0077	2.7913	0.0458
		β	0.9651	0.0081	1.1266	0.0123
		c	1.2695	0.0095	1.3347	0.0315
		k	1.5136	0.0091	1.6052	0.0515
	80	α	2.6897	0.0023	2.7834	0.0450
		β	1.2649	0.0099	1.3612	0.0153
		c	1.3146	0.0085	1.4203	0.0300
		k	1.4591	0.0079	1.6073	0.0473
	100	α	2.4454	0.0022	2.6099	0.0380
		β	1.4999	0.0182	1.6572	0.0343
		c	1.2882	0.0084	1.3883	0.0314
		k	1.5821	0.0008	1.4503	0.0382

Table 12: Bayes estimates and the corresponding PRs via the LINEX loss function with ($\nu = 0.5, 1.5$), different sample sizes n, under Scheme IV and based on Priors I and II

n	m	$\underline{\theta}$	$\nu = 0.5$				$\nu = 1.5$			
			Prior I		Prior II		Prior I		Prior II	
			Est.	PR	Est.	PR	Est.	PR	Est.	PR
50	30	α	2.6065	0.0025	2.6629	0.0131	2.6014	0.0077	2.6382	0.0442
		β	0.9210	0.0030	1.1653	0.0074	0.9151	0.0094	1.1514	0.0241
		c	1.2494	0.0023	1.2982	0.0094	1.2447	0.0070	1.2805	0.0306
		k	1.5085	0.0027	1.5357	0.0135	1.5032	0.0081	1.5105	0.0456
	40	α	2.5892	0.0024	2.6553	0.0127	2.5846	0.0075	2.6312	0.0424
		β	0.9645	0.0028	1.1427	0.0063	0.9590	0.0088	1.1312	0.0205
		c	1.2487	0.0022	1.3056	0.0098	1.2444	0.0066	1.2869	0.0321
		k	1.4903	0.0025	1.5256	0.0121	1.4855	0.0076	1.5031	0.0408
	50	α	2.5486	0.0022	2.5616	0.0102	2.5442	0.0067	2.5418	0.0332
		β	1.0751	0.0042	1.1601	0.0059	1.0670	0.0132	1.1493	0.0190
		c	1.2469	0.0020	1.2645	0.0097	1.2428	0.0062	1.2464	0.0319
		k	1.4740	0.0021	1.4655	0.0120	1.4699	0.0065	1.4428	0.0402
60	36	α	2.5900	0.0034	2.6968	0.0122	2.5834	0.0106	2.6726	0.0403
		β	0.9024	0.0027	0.9572	0.0038	0.8972	0.0085	0.9499	0.0121
		c	1.2357	0.0022	1.2488	0.0083	1.2314	0.0067	1.2331	0.0271
		k	1.5018	0.0023	1.5716	0.0126	1.4973	0.0071	1.5474	0.0423
	48	α	2.6020	0.0023	2.6786	0.0119	2.5974	0.0071	2.6555	0.0392
		β	1.0287	0.0037	1.0970	0.0048	1.0216	0.0114	1.0880	0.0152
		c	1.2573	0.0020	1.2555	0.0093	1.2532	0.0063	1.2384	0.0309
		k	1.5000	0.0024	1.5386	0.0124	1.4952	0.0073	1.5153	0.0416
	60	α	2.4901	0.0010	2.5448	0.0112	2.4881	0.0030	2.5233	0.0375
		β	1.6332	0.0057	1.5478	0.0099	1.6219	0.0181	1.5294	0.0328
		c	1.3379	0.0010	1.3930	0.0095	1.3359	0.0029	1.3751	0.0311
		k	1.4566	0.0017	1.4411	0.0103	1.4531	0.0053	1.4216	0.0338
100	60	α	2.6172	0.0019	2.7799	0.0117	2.6133	0.0060	2.7576	0.0388
		β	0.9631	0.0020	1.1235	0.0031	0.9591	0.0062	1.1175	0.0096
		c	1.2672	0.0024	1.3270	0.0081	1.2624	0.0072	1.3125	0.0270
		k	1.5114	0.0023	1.5925	0.0133	1.5068	0.0070	1.5683	0.0453
	80	α	2.6892	0.0006	2.7722	0.0114	2.6880	0.0017	2.7500	0.0376
		β	1.2624	0.0025	1.3574	0.0039	1.2575	0.0076	1.3500	0.0122
		c	1.3124	0.0022	1.4129	0.0076	1.3083	0.0066	1.3986	0.0248
		k	1.4572	0.0020	1.5956	0.0120	1.4532	0.0061	1.5723	0.0400
	100	α	2.4449	0.0006	2.6005	0.0097	2.4438	0.0017	2.5824	0.0321
		β	1.4953	0.0046	1.6488	0.0088	1.4865	0.0145	1.6330	0.0296
		c	1.2861	0.0021	1.3806	0.0081	1.2820	0.0064	1.3663	0.0271
		k	1.5819	0.0002	1.4410	0.0098	1.5815	0.0006	1.4231	0.0330

Table 13: 95% credible intervals along with their width for different sample sizes n, under Scheme IV and based on Prior I

n	m	$\underline{\theta}$	Prior I		
			Upper	Lower	Width
50	30	α	2.8646	2.4055	0.4591
		β	1.1709	0.6331	0.5379
		c	1.4458	1.0492	0.3967
		k	1.7077	1.3158	0.3919
	40	α	2.8002	2.3975	0.4027
		β	1.1709	0.7633	0.4076
		c	1.4525	1.0640	0.3886
		k	1.6967	1.3095	0.3872
	50	α	2.7534	2.3674	0.3861
		β	1.2035	0.7602	0.4433
		c	1.4710	1.0937	0.3773
		k	1.6617	1.2761	0.3855
60	36	α	2.7837	2.3962	0.3874
		β	1.1259	0.7202	0.4057
		c	1.4802	1.1117	0.3684
		k	1.6910	1.3008	0.3902
	48	α	2.7922	2.4105	0.3817
		β	1.2031	0.8017	0.4014
		c	1.4571	1.0920	0.3651
		k	1.6834	1.3009	0.3825
	60	α	2.7384	2.3615	0.3769
		β	1.4010	0.9467	0.4543
		c	1.3496	0.9929	0.3567
		k	1.6618	1.2860	0.3758
100	60	α	2.7954	2.4203	0.3751
		β	1.2046	0.8515	0.3531
		c	1.4621	1.1085	0.3536
		k	1.7081	1.3251	0.3830
	80	α	2.7739	2.4054	0.3685
		β	1.3331	0.9819	0.3512
		c	1.4114	1.0907	0.3208
		k	1.6859	1.3093	0.3766
	100	α	2.7153	2.3756	0.3396
		β	1.5133	1.0184	0.4949
		c	1.4422	1.1252	0.3170
		k	1.5763	1.2764	0.2999

Table 14: The K-S, A-D, C-V test, along with the tests' respective p-values (shown in square brackets), AIC, BIC, and CAIC

K-S (P-value)	A-D (P-value)	C-V (P-value)	AIC	BIC	CAIC
0.0943 (0.9747)	0.4040 (0.8444)	0.0462 (0.8990)	286.116	293.997	286.95

Table 15: The K-S, A-D, C-V test, along with the tests' respective p-values (shown in square brackets), AIC, BIC, and CAIC for China adaptive Type II progressively censored data

K-S (P-value)	A-D (P-value)	C-V (P-value)	AIC	BIC	CAIC
0.1293 (0.6589)	0.3112 (0.9290)	0.0495 (0.8827)	80.1168	85.9797	81.5983

Table 16: The parameters Bayes estimates of the C-BXII distribution and their PRs, and the 95% credible intervals with their widths

$\underline{\theta}$	SE		LINEX				95% CI		
	Est.	PR	$\nu = 0.5$		$\nu = 1.5$		UL	LL	Width
			Est.	PR	Est.	PR			
α	0.3845	0.0040	0.3825	0.0010	0.3805	0.0031	0.5129	0.2691	0.2438
β	0.4566	0.0061	0.4551	0.0015	0.4521	0.0046	0.6134	0.3045	0.3089
c	1.5061	0.0109	1.5034	0.0027	1.4980	0.0084	1.7350	1.3114	0.4236
k	0.4583	0.0084	0.4562	0.0021	0.4521	0.0065	0.6521	0.3010	0.3511

5. Conclusion

This paper contributes to parameter estimation for the C-BXII distribution through a Bayesian approach incorporating an adaptive Type II progressive censoring approach. A joint posterior distribution based on independent informative gamma priors was developed, and an adaptive Metropolis algorithm was implemented for Bayesian estimation using the SE loss function as a symmetric loss function, and the LINEX loss function as an asymmetric alternative with shape parameters 0.5 and 1.5. The performance of the resulting estimators is evaluated via a simulation study involving four censoring schemes, assessing Bayes estimates, PRs, and 95% CI widths. The results demonstrate that estimation efficiency improves with larger sample sizes and more observed failures. Furthermore, the LINEX loss with a smaller shape parameter yields higher efficiency compared to a larger shape parameter, and it generally outperforms the SE loss. We also show that reducing the prior variance improves the efficiency of the Bayes estimates.

The findings of this study offer valuable implications for both industrial and biomedical applications. In life-testing experiments, the proposed adaptive censoring framework allows for more efficient test designs by enabling the observation of more failures with fewer test units, ultimately improving reliability estimates while conserving resources. In epidemiological contexts, particularly for diseases such as COVID-19, the proposed Bayesian approach with flexible loss functions enables more accurate modeling of survival data, better risk assessment, and informed decision-making in public health. The combination of adaptive censoring, flexible loss functions, and efficient MCMC estimation provides an effective modeling approach for real-world survival analysis under censoring constraints. A comprehensive comparison between the proposed Bayesian estimation method and non-Bayesian approaches has been identified as a promising direction for future research.

Such an investigation would provide further insights into the relative performance and applicability of the competing methodologies within the context of the C-BXII model.

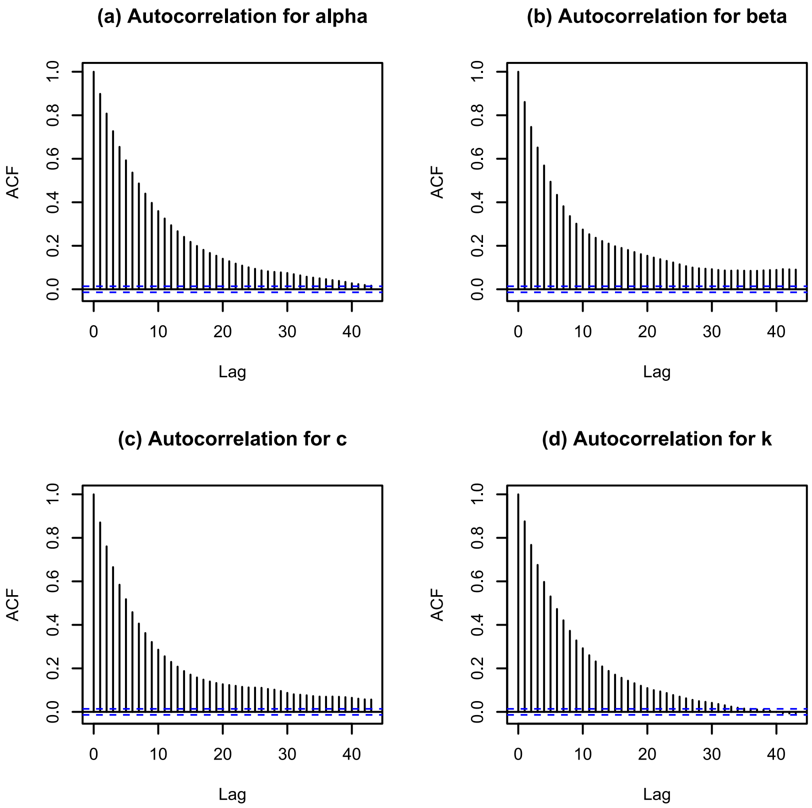

Fig. 17: Autocorrelation plots of α, β, c and k for the application

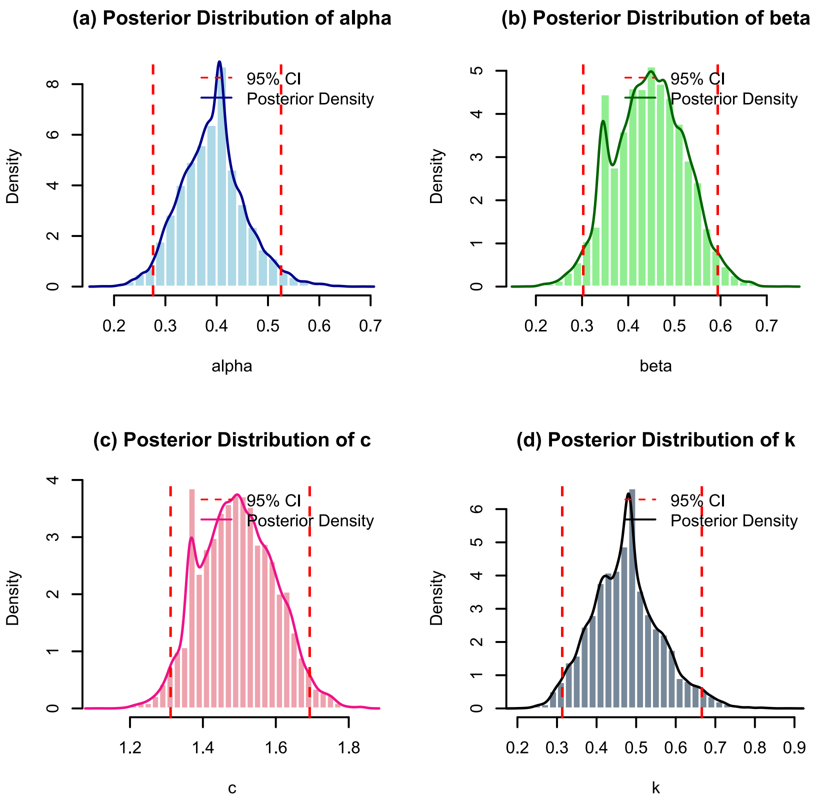

Fig. 18: Histogram, posterior density, and the 95% credible intervals for C-BXII distribution's parameters

List of abbreviations

-2L	-2 log-likelihood
ABXII	Additive Burr XII
AC	Additive Chen
AC-P	Additive Chen–Perks
A-D	Anderson–Darling
AIC	Akaike information criterion
AM	Adaptive Metropolis
AR	Acceptance rate
AW	Additive Weibull
BIC	Bayesian information criterion
BXII	Burr XII
BXII-MW	Burr XII modified Weibull
C	Chen
C-BXII	Chen–Burr XII
CAIC	Corrected Akaike information criterion
C-V	Cramér–von Mises
Cdf	Cumulative distribution function
CI	Credible interval
hrf	Hazard rate function
K-S	Kolmogorov–Smirnov
LF	Likelihood function
LINEX	Linear-exponential
LL	Lower limit
MCMC	Markov Chain Monte Carlo
ML	Maximum likelihood
pdf	Probability density function
PR	Posterior risk
rf	Reliability function
SE	Squared error
UL	Upper limit
W	Weibull
W-C	Weibull–Chen

List of symbols

$\underline{\theta}$	Vector of C-BXII's parameters
ψ	An unknown parameter
$\tilde{\psi}$	Bayes estimator
ν	Shape parameter of the LINEX loss function
$(R_1, R_2, \cdots, R_m)$	Progressive scheme
X_m	Time of the m^{th} failure
T	Experiment terminating time
$L(\underline{\theta}\|\underline{x})$	Likelihood function for C-BXII distribution
$\pi(\underline{\theta})$	Joint prior distribution
$\pi(\underline{\theta}\|\underline{x})$	Joint posterior distribution
B	Normalizing constant
$\pi_i(\theta_i\|\underline{x})$	Marginal posterior distribution of θ_i
$\tilde{\theta}_{SE}$	Bayes estimator under the SE loss function
$PR(\tilde{\theta}_{SE})$	PR of Bayes estimator under the SE loss function
$\tilde{R}_{SE}(x_0)$	Bayes estimator of the rf under the SE loss function
$\tilde{h}_{SE}(x_0)$	Bayes estimator of the hrf under the SE loss function
$\tilde{r}_{SE}(x_0)$	Bayes estimator of the rhrf under the SE loss function
$\tilde{\theta}_{LIN}$	Bayes estimator under the LINEX loss function
$PR(\tilde{\theta}_{LIN})$	PR of Bayes estimator under the LINEX loss function
$\tilde{R}_{LIN}(x_0)$	Bayes estimator of the rf under the LINEX loss function
$\tilde{h}_{LIN}(x_0)$	Bayes estimator of the hrf under the LINEX loss function
$\tilde{r}_{LIN}(x_0)$	Bayes estimator of the rhrf under the LINEX loss function
$L_i(\underline{x})$	LL of the 95% CI for θ_i
$U_i(\underline{x})$	UL of the 95% CI for θ_i
C_ξ	Covariance matrix
M	Burn-in period
h	Total number of iterations

Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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