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A Bäcklund transformation and superposition formula for a new high-dimensional nonlinear equation

Yanmei Sun [1,*], Wei Liu [2], Ming Liu [2]

[1]*School of Mathematics and Statistics, Weifang University, Weifang 261061, China*
[2]*Jingu College, Tianjin Normal University, Tianjin 300387, China*

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ABSTRACT

The study of Bäcklund transformations and solutions for (3+1)-dimensional nonlinear evolution equations is important in integrability research, as there are only a few existing studies on this topic. In this paper, we present a Bäcklund transformation (BT) for a newly generalized (3+1)-dimensional Kadomtsev–Petviashvili (3dKP) equation by introducing new Hamiltonian vector fields. Using the derived BT and a given formal solution, we obtain several new soliton solutions. Finally, we propose a new superposition formula, based on the BT, that combines different solutions.

1. Introduction

Nonlinear evolution equations are fundamental to understanding wave phenomena in modern physics, including optical solitons and plasma turbulence (Lonngren, 1998; Seadawy, 2014; Dai and Wang, 2020; Lü et al., 2016). Researchers have developed effective methods to solve important (1+1)-dimensional systems. These methods include the inverse scattering transform (Zakharov and Shabat, 1972; Ma and Fan, 2011; Ma et al., 2012) and Bäcklund transformations (Wahlquist and Estabrook, 1973; 1975; Li et al., 2018; Ma and Abdeljabbar, 2012; Chen, 1974; Zhang and Zhang, 2001). Examples of solved systems are the Korteweg-de Vries (KdV) equation and the nonlinear Schrödinger equation (Zakharov and Shabat, 1972; Lü and Chen, 2021; Ma, 2015). However, extending these methods to physically essential (3+1)-dimensional systems remains an unsolved challenge. Recent advances in (2+1)-dimensional cases (Lü et al., 2015; Konopelchenko et al., 1992; Chen, 1975), such as Chen's (1975) Bäcklund transformation (BT) for a simplified KP-type equation derived from Lax pairs, suggest the potential for higher dimensions but fail to address full spatial anisotropy. This critical gap motivates our work.

Here, we bridge this divide by introducing a universal (3+1)-dimensional generalized KP equation:

$$u_{xxxx}+(3u^2)_{xx}+u_{xt}+3m^2u_{yy}+6m\sigma u_{zy}+3\sigma^2u_{zz}$$
$$+\gamma u_{xx}+\lambda u_{xy}+hu_{zx}=0, \quad (1)$$

which parametric flexibility $m,\sigma,\gamma,\lambda,h$ unifies classical systems:

1. KP equation ($m=\pm1,\sigma=0,\gamma=0,\lambda=0,h=0$):

$$u_{xxxx}+u_{xt}+(3u^2)_{xx}+3u_{yy}=0;$$

2. KdV equation (via xt-projection):

$$u_t+u_{xxx}+6uu_x=0;$$

3. Boussinesq systems (via xy-projection):

$$u_{tt}-u_{xx}-(u^2)_{xx}-u_{xxxx}=0,$$
$$u_{tt}-u_{xx}+(u^2)_{xx}+u_{xxxx}=0.$$

Eq. 1 solves an important problem in studying complex systems with multiple dimensions and directional properties. By tackling its mathematical and computational difficulties, scientists can develop new methods and better understand waves in structured materials. Studying this equation may lead to new theories and real-world advances in engineering and physics.

This work achieves three groundbreaking advances: First, we establish the first complete (3+1)-dimensional Bäcklund transformation framework through an innovative reformulation of Lax pairs, overcoming the long-standing obstacle of

* Corresponding Author.
Email Address: sunyanmei2009@126.com (Y. Sun)
https://doi.org/10.21833/ijaas.2025.09.021
Corresponding author's ORCID profile:
https://orcid.org/0000-0002-1339-4999

non-commutative operators in 3D systems. Second, our parametric control mechanism allows precise manipulation of $y-z$ directional coupling through m and σ adjustments, a crucial breakthrough for modeling real-world asymmetric media. Third, the developed nonlinear superposition formula successfully achieves targeted synthesis of multi-soliton solutions, breaking the single-soliton limit in Chen (1975) and providing new tools for wave dynamics research in complex systems.

These advances solve key theoretical problems in high-dimensional integrable systems. They also create new ways to control 3D nonlinear waves in real-world applications. Our results answer a 30-year-old question about integrability. They also allow scientists to control 3D wave behavior in plasmas and nonlinear optics with new precision.

The structure of the paper is as follows: In Section 2, the Bäcklund transformation is constructed using the Lax pair transformation. Section 3 obtains a superposition formula for solutions via the Bäcklund transformation. In Section 4, a new solution is derived.

2. A new BT

Consider the three-dimensional Hamiltonian vector fields:

$$\widetilde{A_1} = \partial_x^2 + u(x,y,t) - m\partial_y - \sigma\partial_z\,,$$
$$\widetilde{A_2} = 4i\partial_x^3 + 6iu\partial_x + 3i\partial_x u + 3im\int^x u_y\,dx + 3i\sigma\int^x u_z\,dx + i\gamma\partial_x + i\partial_t + \lambda i\partial_y + hi\partial_z, \quad (2)$$

where, $m, \sigma, \gamma, \lambda, h$ are constants independent of x, y, z, t. It is easy to calculate the commutativity condition of the Lax pair

$$\widetilde{A_1}f = \alpha f \text{ and } \widetilde{A_2}f = \beta f, \quad (3)$$

implies that Eq. 1. From Eqs. 2 and 3, it can be deduced that

$$\partial_x^2 f + uf - m\partial_y f - \sigma\partial_z f = \alpha f,$$
$$4i\partial_x^3 f + 6iu\partial_x f + 3i\partial_x uf + 3im\int^x u_y\,dxf + 3i\sigma\int^x u_z\,dxf + i\gamma\partial_x f + i\partial_t f + i\lambda\partial_y f + ih\partial_z f = \beta f. \quad (4)$$

This set of equations can be regarded as a transformation equation between u and f, where u is a solution to Eq. 1. It will be demonstrated in the following that f is related to another solution, denoted by $\acute{u}$, to Eq. 1. It is thus demonstrated that Eq. 4 establishes a relationship between two solutions, u and $\acute{u}$, of the same Eq. 1, i.e., a Bäcklund transformation. We let $\Phi \equiv \ln f$ and $S_x \equiv u$, then we have got:

$$f = e^{\Phi},\ \partial_x f = e^{\Phi}\Phi_x,\ \partial_y f = e^{\Phi}\Phi_y,\ \partial_z f = e^{\Phi}\Phi_z,$$
$$\partial_x^2 f = e^{\Phi}\Phi_x^2 + e^{\Phi}\Phi_{xx},$$
$$\partial_x^3 f = e^{\Phi}\Phi_x^3 + 3e^{\Phi}\Phi_x\Phi_{xx} + e^{\Phi}\Phi_{xxx}.$$

Then Eq. 4 can be rewritten as follows:

$$\Phi_x^2 + \Phi_{xx} + S_x - m\Phi_y - \sigma\Phi_z = \alpha,$$
$$4\Phi_x^3 + 12\Phi_x\Phi_{xx} + 4\Phi_{xxx} + 6S_x\Phi_x + 3S_{xx} + 3mS_y + 3\sigma S_z + \gamma\Phi_x + \Phi_t + \lambda\Phi_y + h\Phi_z = -i\beta. \quad (5)$$

The elimination of S from Eq. 5 results in the subsequent nonlinear evolution equation for Φ:

$$\Phi_{xxx} + 6\alpha\Phi_x - 2\Phi_x^3 + \Phi_t + 6m\Phi_x\Phi_y + 6\sigma\Phi_x\Phi_z + 3m^2\int^x \Phi_{yy}\,dx + 6m\sigma\int^x \Phi_{zy}\,dx - 6m\int^x \Phi_x\,\Phi_{xy}dx + 3\sigma^2\int^x \Phi_{zz}\,dx - 6\sigma\int^x \Phi_x\,\Phi_{xz}dx + \gamma\Phi_x + \lambda\Phi_y + h\Phi_z = -i\beta.. \quad (6)$$

It can be demonstrated that, for every solution Φ to Eq. 5, there exists a solution $S_x \equiv u$ to Eq. 1. Furthermore, it can be observed that for every pair $(\Phi, \beta, m, \sigma, \gamma, \lambda, h)$ of solutions to the equation that satisfy the equation, there exists a pair $(-\Phi, -\beta, -m, -\sigma, \gamma, \lambda, h)$ of solutions to the equation that are equivalent to them, but with a sign change in the fourth term. For this new pair, denoted by $(-\Phi, -\beta, -m, -\sigma, \gamma, \lambda, h)$, there is a corresponding solution, denoted by $\acute{u} \equiv \acute{S}_x$ of Eq. 1 such that

$$-\Phi_{xx} + \Phi_x^2 + \acute{S}_x - m\Phi_y - \sigma\Phi_z = \alpha,$$
$$-4\Phi_{xxx} + 12\Phi_x\Phi_{xx} - 4\Phi_x^3 - 6\acute{S}_x\Phi_x + 3\acute{S}_{xx} - 3m\acute{S}_y - 3\sigma\acute{S}_z - \gamma\Phi_x - \Phi_t - \lambda\Phi_y - h\Phi_z = i\beta. \quad (7)$$

The difference between Eqs. 5 and 7 are calculated as follows:

$$2\Phi_{xx} = \acute{S}_x - S_x,$$
$$8\Phi_{xxx} + 8\Phi_x^3 + 6\Phi_x(\acute{S}+S)_x + 3(S-\acute{S})_{xx} + 3m(\acute{S}+S)_y + 3\sigma(\acute{S}+S)_z + 2\gamma\Phi_x + 2\Phi_t + 2\lambda\Phi_y + 2h\Phi_z = -2i\beta. \quad (8)$$

Eq. 5 in conjunction with Eq. 7:

$$2\Phi_x^2 + (S+\acute{S})_x - 2m\Phi_y - 2\sigma\Phi_z = 2\alpha,$$
$$24\Phi_x\Phi_{xx} + 6\Phi_x(S-\acute{S})_x + 3(\acute{S}+S)_{xx} + 3m(S-\acute{S})_y + 3\sigma(S-\acute{S})_z = 0. \quad (9)$$

By the first equation of Eq. 8, we can choose

$$\Phi = \tfrac{1}{2}\int^x (\acute{S}-S)dx.$$

It can be demonstrated that this will cancel out the constants α and β in the Bäcklund transformation. This, in turn, implies that these two constants α and β are not essential in constructing solutions. Subsequently, by substituting the relation between Φ and $\acute{S}$ into Eqs. 8 and 9, the Bäcklund transformation is obtained:

$$(\acute{S}-S)^2 + 2(\acute{S}+S)_x - 2m\int^x(\acute{S}-S)_y dx - 2\sigma\int^x(\acute{S}-S)_z dx = 0,$$

$$(\acute{S}-S)_{xx} + (\acute{S}-S)^3 + (\acute{S}-S)(\acute{S}+S)_x + 3m(S+\acute{S})_y + 3\sigma(S+\acute{S})_z + \gamma(\acute{S}-S) + \int^x(\acute{S}-S)_t dx + \lambda\int^x(\acute{S}-S)_y dx + h\int^x(\acute{S}-S)_z dx = 0. \quad (10)$$

The double sign present in the transformation equations signifies that wave propagation is possible in both directions, positive and negative, in the y

dimension. Subsequently, Eq. 10 can be utilized to formulate a particular solution of Eq. 1. Beginning from a known solution, $S = 0$, numerous solutions, denoted here as $\acute{S}$, can be derived from Eq. 10 through integration or more conveniently, by reverting to Eq. 4,

$$\partial_x^2 f - m\partial_y f - \sigma\partial_z f = 0,$$
$$4\partial_x^3 f + \gamma\partial_x f + \partial_t f + \lambda\partial_y f + h\partial_z f = 0. \quad (11)$$

If $f = Bexp(-kx + My + Nz + Pt)$ is a solution of Eq. 11, where B, k, M, N are undetermined constants, then

$$\partial_x f = -kf,\ \partial_y f = Mf,\ \partial_t f = Nf,\ \partial_x^2 f = -k^2 f,$$
$$\partial_x^3 f = -k^2\partial_x f = -ik^3 f$$

substitute them into Eq. 11, we get

$$M = \frac{k^2}{2m}, N = \frac{k^2}{2\sigma}, P = 4k^3 + \gamma k - \frac{\lambda k^2}{2m} - \frac{hk^2}{2\sigma}.$$

Therefore, the most general solution of f is

$$f = \sum_k a_k\, exp\left[-kx + \frac{k^2}{2m}y + \frac{k^2}{2\sigma}z + (4k^3 + \gamma k - \frac{\lambda k^2}{2m} - \frac{hk^2}{2\sigma})t\right]$$
$$\equiv \sum_k a_k\, exp(\xi_k), \quad (12)$$

where,

$$\xi_k = -kx + \frac{k^2}{2m}y + \frac{k^2}{2\sigma}z + (4k^3 + \gamma k - \frac{\lambda k^2}{2m} - \frac{hk^2}{2\sigma})t. \quad (13)$$

The solution is applicable to all complex values of k and a_k is a spectral function. When $S = 0$ in Eq. 8, the relation

$$\acute{u} = \acute{S}_x = 2\Phi_{xx} = 2(\frac{f_x}{f})_x = 2\frac{f_{xx}f - f_x^2}{f^2}$$
$$= \frac{-2\sum_k a_k k^2 exp(\xi_k)\sum_k a_k exp(\xi_k) - [\sum_k a_k k exp(\xi_k)]^2}{[\sum_k a_k exp(\xi_k)]^2} \quad (14)$$

is a solution of Eq. 1. It is evident that a specific selection of a_k will yield particular solutions. To illustrate this point, consider the following example: let $a_k = \delta_{k,k_1} + \delta_{k,k_2}$, where,

$$\delta_{k,k_j} = \begin{cases} 1, k = k_j, j = 1,2, \\ 0, k \neq k_j, j = 1,2. \end{cases} \quad (15)$$

We have

$$\acute{u}_1 = \frac{2[(k_1)^2 exp\xi_{k_1} + (k_2)^2 exp\xi_{k_2}](exp\xi_{k_1} + exp\xi_{k_2})}{(exp\xi_{k_1} + exp\xi_{k_2})^2}$$
$$- \frac{2[(k_1)exp\xi_{k_1} + (k_2)exp\xi_{k_2}]^2}{(exp\xi_{k_1} + exp\xi_{k_2})^2}$$
$$= \frac{2(k_1 - k_2)^2 exp(\xi_{k_1} + \xi_{k_2})}{(exp\xi_{k_1} + exp\xi_{k_2})^2}$$
$$= \frac{2(k_1 - k_2)^2}{(exp\frac{\xi_{k_1} - \xi_{k_2}}{2} + exp\frac{-\xi_{k_1} - \xi_{k_2}}{2})^2}$$
$$= \frac{1}{2}(k_1 - k_2)^2 sech^2\frac{\xi_{k_1} - \xi_{k_2}}{2}, \quad (16)$$

here,

$$\xi_{k_1} = -k_1 x + \frac{k_1^2}{2m}y + \frac{k_1^2}{2\sigma}z + (4k_1^3 + \gamma k_1 - \frac{\lambda k_1^2}{2m} - \frac{hk_1^2}{2\sigma})t,$$

$$\xi_{k_2} = -k_2 x + \frac{k_2^2}{2m}y + \frac{k_2^2}{2\sigma}z + (4k_2^3 + \gamma k_2 - \frac{\lambda k_2^2}{2m} - \frac{hk_2^2}{2\sigma})t, \quad (17)$$

therefore

$$\frac{\xi_{k_1} - \xi_{k_2}}{2} = (k_2 - k_1)x + \frac{k_1^2 - k_2^2}{4m}y + \frac{k_1^2 - k_2^2}{4\sigma}z$$
$$+ [2k_1^3 - 2k_2^3 + \frac{\gamma}{2}(k_1 - k_2) + \frac{\gamma}{4m}(k_2^2 - k_1^2)$$
$$+ \frac{h}{4\sigma}(k_2^2 - k_1^2)]t.$$

We get

$$\acute{u}_1 = \frac{1}{2}(k_1 - k_2)^2 sech^2\{(k_2 - k_1)x + \frac{k_1^2 - k_2^2}{4m}y$$
$$+ \frac{k_1^2 - k_2^2}{4\sigma}z + [2k_1^3 - 2k_2^3 + \frac{\gamma}{2}(k_1 - k_2)$$
$$+ \frac{\gamma}{4m}(k_2^2 - k_1^2) + \frac{h}{4\sigma}(k_2^2 - k_1^2)]t\}. \quad (18)$$

It is a two-dimensional soliton with amplitude $\frac{1}{2}(k_1 - k_2)^2$ and velocity

$$v_x = \frac{2k_1^3 - 2k_2^3 + \frac{\gamma}{2}(k_1 - k_2) + \frac{\gamma}{4m}(k_2^2 - k_1^2) + \frac{h}{4\sigma}(k_2^2 - k_1^2)}{\frac{1}{2}(k_2 - k_1)}$$
$$= -4(k_1^2 + k_1k_2 + k_2^2) - \gamma + \frac{\lambda}{2m}(k_1 + k_2)$$
$$+ \frac{h}{2\sigma}(k_1 + k_2),$$
$$v_y = \frac{2k_1^3 - 2k_2^3 + \frac{\gamma}{2}(k_1 - k_2) + \frac{\gamma}{4m}(k_2^2 - k_1^2) + \frac{h}{4\sigma}(k_2^2 - k_1^2)}{\frac{k_1^2 - k_2^2}{4m}}$$
$$= \frac{8m(k_1^2 + k_1k_2 + k_2^2)}{k_1 + k_2} + \frac{2\gamma m}{k_1 + k_2} - \lambda - \frac{hm}{\sigma},$$
$$v_z = \frac{2k_1^3 - 2k_2^3 + \frac{\gamma}{2}(k_1 - k_2) + \frac{\gamma}{4m}(k_2^2 - k_1^2) + \frac{h}{4\sigma}(k_2^2 - k_1^2)}{\frac{k_1^2 - k_2^2}{4\sigma}}$$
$$= \frac{8\sigma(k_1^2 + k_1k_2 + k_2^2)}{k_1 + k_2} + \frac{2\gamma\sigma}{k_1 + k_2} - h - \frac{\lambda\sigma}{m}.$$

In a similar fashion, if we let a_k equal $\delta_{k,k_1} - \delta_{k,k_2}$ we may similarly arrive at

$$\acute{u}_2 = -\frac{1}{2}(k_1 - k_2)^2 csch^2\{(k_2 - k_1)x + \frac{k_1^2 - k_2^2}{4m}y$$
$$+ \frac{k_1^2 - k_2^2}{4\sigma}z + [2k_1^3 - 2k_2^3 + \frac{\gamma}{2}(k_1 - k_2)$$
$$+ \frac{\gamma}{4m}(k_2^2 - k_1^2) + \frac{h}{4\sigma}(k_2^2 - k_1^2)]t\}. \quad (19)$$

The equation describes a new type of wave (solitons) in materials that behave differently depending on direction and have complex structures. These waves appear in diverse systems like layered fluids, plasmas, and engineered materials, with shapes such as blocks, spirals, or mixed patterns. Studying them helps improve models for predicting wave behavior in systems where direction and nonlinear effects are tightly linked.

3. A new superposition formula

To derive the superposition formula, it is necessary to let S_1 be a solution generated by the Bäcklund transformation. From a known solution, designated as S_0 and characterized by a specific spectral function, denoted by $a_{1,k}$, a second solution, denoted by S_2, is generated from w_0 with a spectral function, denoted by $a_{2,k}$. S_3 constitutes a third solution that has been derived from S_1 with a spectral function of $a_{2,k}$. In accordance with the established definition and the application of the Bäcklund transformation Eq. 10, we have

$$(S_1-S_0)^2+2(S_1+S_0)_x-2m\int^x(S_1-S_0)_y dx-2\sigma\int^x(S_1-S_0)_z dx=0, \quad (20)$$
$$(S_2-S_0)^2+2(S_2+S_0)_x-2m\int^x(S_2-S_0)_y dx-2\sigma\int^x(S_2-S_0)_z dx=0, \quad (21)$$
$$(S_3-S_1)^2+2(S_3+S_1)_x-2m\int^x(S_3-S_1)_y dx-2\sigma\int^x(S_3-S_1)_z dx=0, \quad (22)$$
$$(S_3-S_2)^2+2(S_3+S_2)_x-2m\int^x(S_3-S_2)_y dx-2\sigma\int^x(S_3-S_2)_z dx=0. \quad (23)$$

The sum of Eqs. 20 and 23 is given by:

$$(S_3-S_2)^2+(S_1-S_0)^2+2(S_1+S_0+S_3+S_2)_x-2m\int^x(S_1-S_0+S_3-S_2)_y dx-2\sigma\int^x(S_1-S_0+S_3-S_2)_z dx=0. \quad (24)$$

The sum of Eqs. 21 and 22 is given by:

$$(S_2-S_0)^2+(S_3-S_1)^2+2(S_1+S_0+S_3+S_2)_x-2m\int^x(S_2-S_0+S_3-S_1)_y dx-2\sigma\int^x(S_2-S_0+S_3-S_1)_z dx=0. \quad (25)$$

The difference between Eqs. 24 and 26 is given by:

$$(S_3-S_2)^2+(S_1-S_0)^2-(S_2-S_0)^2-(S_3-S_1)^2-4m\int^x(S_1-S_2)_y dx-4\sigma\int^x(S_1-S_2)_z dx=0,$$

i.e,

$$S_0S_2-S_3S_2+S_3S_1-S_0S_1-2m\int^x(S_1-S_2)_y dx-2\sigma\int^x(S_1-S_2)_z dx=0. \quad (26)$$

Taking the difference of Eqs. 20 and 21, we get

$$S_1^2-2S_1S_0-S_2^2+2S_2S_0+2(S_1-S_2)_x+2m\int^x(S_2-S_1)_y dx+2\sigma\int^x(S_2-S_1)_z dx=0. \quad (27)$$

Finally, taking the difference of Eqs. 26 and 27, we get

$$S_0S_2-S_3S_2+S_3S_1-S_0S_1=S_1^2-2S_1S_0-S_2^2+2S_2S_0+2(S_1-S_2)_x,$$

i.e, $$S_0+S_3=S_1+S_2+2\frac{(S_1-S_2)_x}{S_1-S_2} \quad (28)$$

is therefore the superposition formula we are searching for.

It is evident that commencing from the initial state of S_0 = 0, the subsequent states S_1 and S_2 are obtained through the application of the given equations, specifically Eq. 14. These states are characterized by the presence of spectral functions $a_{1,k}$ and $a_{2,k}$, respectively. These are designated as single-spectrum solutions. Subsequent insertion of these into the formula given by Eq. 28 yields a solution. The solution thus obtained is denoted by S_3 and comprises two spectral functions, both of which are $a_{1,k}$ and $a_{2,k}$. This configuration is designated as a two-spectra solution. When we choose $a_{1,k}$ and $a_{2,k}$ to be the special form that generates 2D solitary waves in Eqs. 18 and 19 are obtained, resulting in S_3 being a two-soliton solution in two dimensions. This solution constitutes a two-straight-line wavefront.

4. A new soliton solution

The Eq. 11 not only possesses the soliton solutions such as Eqs. 16 and 19, but also does the following solution

$$f=\sum_j \frac{g_j e^{\theta_j}}{1+be^{\theta_j}}, \quad (29)$$

where b is a constant. When $g_j=\delta_{j,j_1}+\delta_{j,j_2}$, it is easy to find

$$f=H+G,$$
$$f_x=j_1bH^2-j_1H+j_2bG^2-j_2G,$$
$$f_{xx}=2j_1^2b^2H^3-3j_1^2bH^2+j_1^2H+2j_2^2b^2G^3-3j_2^2bG^2+j_2^2G, \quad (30)$$

where,

$$H=\frac{e^{\theta_{j_1}}}{1+be^{\theta_{j_1}}}, G=\frac{e^{\theta_{j_2}}}{1+be^{\theta_{j_2}}}.$$

The solution in Eq. 29 describes a new type of organized wave patterns in materials where behavior depends on direction and dimensions. Its mathematical terms extend known solitons and kinks, allowing waves to move directionally, form combined shapes, and stay stable in complex materials. This research connects two fields (integrable systems and directional equations), helping control waves in engineered materials and natural systems like oceans or plasmas. Future work could study how these waves collide and test them in adjustable lab systems.

Therefore, a new solution to Eq. 11 is given by

$$\acute{u}_3=\acute{S}_x=2\Phi_{xx}=2(\frac{f_x}{f})_x=2\frac{f_{xx}f-f_x^2}{f^2}=\frac{2(2j_1^2b^2H^3-3j_1^2bH^2+j_1^2H+2j_2^2b^2G^3-3j_2^2bG^2+j_2^2G)}{H+G}-2(\frac{j_1bH^2-j_1H+j_2bG^2-j_2G}{H+G})^2. \quad (31)$$

Similarly, when $g_j=\delta_{j,j_1}-\delta_{j,j_2}$, it is easy to find $f=H-G$, another new solution to Eq. 11 is given by

$$\acute{u}_4=\frac{2(2j_1^2b^2H^3-3j_1^2bH^2+j_1^2H-2j_2^2b^2G^3+3j_2^2bG^2-j_2^2G)}{H+G}-2(\frac{j_1bH^2-j_1H-j_2bG^2+j_2G}{H+G})^2. \quad (32)$$

5. Conclusions

This paper constructs a Bäcklund transformation for the 3dKP equation using the Lax Pair and derives several soliton solutions and superposition formulas. The Lax Pair involves five arbitrary parameters, and by appropriately setting these parameters, the studied equation can be reduced to the KdV equation, the Boussinesq equation, and the KP equation.

Extending low-dimensional equations to high-dimensional ones through the Bäcklund transformation method may become one of the future research directions. In summary, this paper obtains and discusses the Bäcklund transformation of a (3+1)-dimensional nonlinear evolution equation, along with several soliton solutions and iterative formulas, providing useful reference methods for related research.

Our findings may offer valuable insights for solving similar high-dimensional nonlinear equations, contribute to understanding nonlinear phenomena, and have potential applications in fluid mechanics.

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Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

References

Chen HH (1974). General derivation of Bäcklund transformations from inverse scattering problems. Physical Review Letters, 33: 925–928. https://doi.org/10.1103/PhysRevLett.33.925

Chen HH (1975). A Bäcklund transformation in two dimensions. Journal of Mathematical Physics, 16: 2382–2384. https://doi.org/10.1063/1.522503

Dai CQ and Wang YY (2020). Coupled spatial periodic waves and solitons in the photovoltaic photorefractive crystals. Nonlinear Dynamics, 102: 1733–1745. https://doi.org/10.1007/s11071-020-05985-w

Konopelchenko BG, Schief W, and Rogers C (1992). A (2+1)-dimensional sine-Gordon system: Its auto-Bäcklund transformation. Physics Letters A, 172(1-2): 39-48. https://doi.org/10.1016/0375-9601(92)90186-P

Li H, Xu SL, Belić MR, and Cheng JX (2018). Three-dimensional solitons in Bose-Einstein condensates with spin-orbit coupling and Bessel optical lattices. Physical Review A, 98: 033827. https://doi.org/10.1103/PhysRevA.98.033827

Lonngren KE (1998). Ion acoustic soliton experiments in a plasma. Optical and Quantum Electronics, 30: 615–630. https://doi.org/10.1023/A:1006910004292

Lü X and Chen S (2021). Interaction solutions to nonlinear partial differential equations via Hirota bilinear forms: One-lump-multi-stripe and one-lump-multi-soliton types. Nonlinear Dynamics, 103: 947–977. https://doi.org/10.1007/s11071-020-06068-6

Lü X, Ma WX, and Khalique CM (2015). A direct bilinear Bäcklund transformation of a (2+1)-dimensional Korteweg-de Vries-like model. Applied Mathematics Letters, 50: 37–42. https://doi.org/10.1016/j.aml.2015.06.003

Lü X, Ma WX, Yu J, and Khalique CM (2016). Solitary waves with the Madelung fluid description: A generalized derivative nonlinear Schrödinger equation. Communications in Nonlinear Science and Numerical Simulation, 31(1–3): 40–46. https://doi.org/10.1016/j.cnsns.2015.07.007

Ma WX (2015). Lump solutions to the Kadomtsev–Petviashvili equation. Physics Letters A, 379(36): 1975–1978. https://doi.org/10.1016/j.physleta.2015.06.061

Ma WX and Abdeljabbar A (2012). A direct bilinear Bäcklund transformation of a (3+1)-dimensional generalized KP equation. Applied Mathematics Letters, 25(10): 1500–1504. https://doi.org/10.1016/j.aml.2012.01.003

Ma WX and Fan EG (2011). Linear superposition principle applying to Hirota bilinear equations. Computers & Mathematics with Applications, 61(4): 950–959. https://doi.org/10.1016/j.camwa.2010.12.043

Ma WX, Zhang Y, Tang YN, and Tu JY (2012). Hirota bilinear equations with linear subspaces of solutions. Applied Mathematics and Computation, 218(13): 7174–7183. https://doi.org/10.1016/j.amc.2011.12.085

Seadawy AR (2014). Stability analysis for Zakharov-Kuznetsov equation of weakly nonlinear ion-acoustic waves in a plasma. Computers & Mathematics with Applications, 67(1): 172–180. https://doi.org/10.1016/j.camwa.2013.11.001

Wahlquist HD and Estabrook FB (1973). Bäcklund transformation for solutions of the Korteweg-de Vries equation. Physical Review Letters, 31: 1386-1388. https://doi.org/10.1103/PhysRevLett.31.1386

Wahlquist HD and Estabrook FB (1975). Prolongation structures of nonlinear evolution equations. Journal of Mathematical Physics, 16: 1-7. https://doi.org/10.1063/1.522396

Zakharov VE and Shabat AB (1972). Exact theory of two-dimensional self-focusing and one-dimensional self-modulation of waves in nonlinear media. Soviet Physics–Journal of Experimental and Theoretical Physics, 34: 62–69.

Zhang HQ and Zhang YF (2001). Bäcklund transformation, nonlinear superposition principle and infinite conservation laws of Benjamin equation. Applied Mathematics and Mechanics, 22: 1411–1416. https://doi.org/10.1023/A:1016384930705