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Legendre operational differential matrix for solving ordinary differential equations

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ABSTRACT

In this paper, we used the Legendre operational differential matrix method based on the Tau method to find the approximate analytical solutions to the initial value problems and boundary value problems of ordinary differential equations. This method allows the solution of the ordinary differential equation to be computed in the form of an infinite series in which the components can be easily calculated. We introduced a comparison between the approximate solution that we computed and the exact solution of the selected problem, as we found the absolute error. According to the numerical results, the series of solutions we found are accurate and very close to the exact analytical solutions.

1. Introduction

Many methods have been developed so far for solving ordinary differential equations (ODEs). These methods can provide exact-analytical solutions, approximate-analytical (series) solutions, or numerical solutions for the ODE (Islam, 2015; Hossain et al., 2017; Masenge and Malaki, 2020). The topic of the approximate analytical methods for solving ODE has been rapidly growing in recent years, whereas the series of solutions have been studied by several authors during the past few years (Geng et al., 2009; Sakka and Sulayh, 2019; Moore and Ertürk, 2020). In this work, we have used the Legendre operational differential matrix method based on the Tau method to find the series of solutions for the ODE. This method is a general semi-analytic approach used to obtain a series of solutions for the ODE with initial (or boundary) conditions. The beginning of using this method dates back to 2014, and below, we will introduce a summary of what the researchers who preceded us presented regarding the use of this method.

Jung et al. (2014) used the Legendre operational differential matrix method based on the Tau method to find the approximate analytical solutions of the ODE with initial conditions. Edeo (2019) used the same method to find the approximate analytical solutions of the ODE with boundary conditions. These researchers have solved the following form of the second order ODE:

$$x''(t) + p(t)x'(t) + f(t,x) = g(t) \quad ,t \geq 0 \qquad (1)$$

Of course, Eq. 1 does not include all types of the second-order ODE. Therefore, during this work, we will expand Eq. 1 into the following form, which is a general form that includes all types of the second order ODE:

$$x''(t) = f(t, x(t), x'(t)) \quad ,t \geq 0 \qquad (2)$$

2. Shifted Legendre polynomials

The Legendre polynomials of order r are defined on the interval $[-1,1]$ and are denoted by $L_r(z)$. These polynomials can be determined with the help of the following recurrence relations (Jung et al., 2014):

$$L_0(z) = 1 \qquad (3)$$
$$L_1(z) = z \qquad (4)$$
$$L_2(z) = \frac{3}{2}z^2 - \frac{1}{2} \qquad (5)$$
$$L_3(z) = \frac{5}{2}z^3 - \frac{3}{2}z \qquad (6)$$
$$L_4(z) = \frac{35}{8}z^4 - \frac{15}{4}z^2 + \frac{3}{8} \qquad (7)$$
$$L_{r+1}(z) = \frac{2r+1}{r+1}zL_r(z) - \frac{r}{r+1}L_{r-1}(z) \quad ; \ r = 1,2,3,\ldots. \qquad (8)$$

In order to use the Legendre polynomials on the interval $[0,1]$, the so-called shifted Legendre polynomials are defined by introducing $z = 2t - 1$.

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Let the shifted Legendre polynomials $L_r(2t-1)$ be denoted by $p_r(t)$, then $p_r(t)$ can be obtained as follows:

$$p_0(t) = 1 \quad (9)$$
$$p_1(t) = 2t - 1 \quad (10)$$
$$p_2(t) = 6t^2 - 6t + 1 \quad (11)$$
$$p_3(t) = 20t^3 - 30t^2 + 12t - 1 \quad (12)$$
$$p_4(t) = 70t^4 - 140t^3 + 90t^2 - 20t + 1 \quad (13)$$
$$p_{r+1}(t) = \frac{2r+1}{r+1}(2t-1)p_r(t) - \frac{r}{r+1}p_{r-1}(t) \;;\; r = 1, 2, 3, \quad (14)$$

3. Description of Legendre operational differential matrix method

In order to describe Legendre operational differential matrix method in a simple way, we will consider the following second order initial value problem,

$$x''(t) = f\big(t,\ x(t),\ x'(t)\big) \quad , t \geq 0 \quad (15)$$

with initial conditions:

$$x(0) = a \quad , \quad x'(0) = b. \quad (16)$$

Note that we can deal with the higher-order initial value problem or boundary value problem.

The solution $x(t)$ of problem 15 can be approximated as (Edeo, 2019; Jung et al., 2014):

$$x(t) = \sum_{r=0}^{\infty} c_r\, p_r(t) \quad (17)$$

where, $p_r(t)$ are the shifted Legendre polynomials, c_r are the shifted Legendre coefficients.

Also, the coefficients c_r are given by:

$$c_r = (2r+1)\int_0^1 x(t)\, p_r(t)\, dt \quad ; r = 0,1,2,\ldots \quad (18)$$

Finding the approximate solution $x(t)$ depends mainly on finding the constants c_r as we will notice later.

By considering the first (m+1) terms of the series solution (Eq. 17), we get:

$$x(t) \approx \sum_{r=0}^{m} c_r\, p_r(t) \quad (19)$$

and that gives:

$$x(t) \approx c_0\, p_0(t) + c_1\, p_1(t) + c_2\, p_2(t) + \cdots + c_m\, p_m(t) \quad (20)$$

and in matrix form, we can get:

$$x(t) \approx C^T W(t) \quad (21)$$

where, $C^T = [c_0, c_1, \ldots, c_m]$ is the shifted Legendre coefficients, $W(t) = [p_0(t), p_1(t), \ldots, p_m(t)]^T$ is the shifted Legendre vector.

The derivative of the vector $W(t)$ can be expressed as:

$$\frac{d\,W(t)}{dt} = D^{(1)}\, W(t) \quad (22)$$

where, $D^{(1)}$ is $(m+1) \times (m+1)$ operational differential matrix, which is given by:

$$D^{(1)} = (d_{ij}) = \begin{cases} 4j-2\,, & if\ j = i-k \\ 0 & ,otherwise \end{cases} \quad (23)$$

where,

$$k = \begin{cases} 1,3,5,\ldots m & if\ \ m\ is\ odd \\ 1,3,5,\ldots m-1 & if\ m\ is\ even \end{cases} \quad (24)$$

In this work, we will consider $m = 4$, then we get:

$$D^{(1)} = (d_{ij}) = \begin{cases} 4j-2\,, & if\ j = i-1\ or\ j = i-3 \\ 0\,, & otherwise \end{cases} \quad (25)$$

therefore, the operational differential matrix will be:

$$D^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 \end{bmatrix} \quad (26)$$

for the nth order derivative:

$$\frac{d^n W(t)}{dt^n} = \left(D^{(1)}\right)^n W(t) = D^{(n)} W(t) \quad ; \quad n = 1, 2, 3, \ldots \quad (27)$$

where, $\left(D^{(1)}\right)^n$ denotes the matrix powers. Thus, we can find:

$$D^{(2)} = D^{(1)} \times D^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 \\ 40 & 0 & 140 & 0 & 0 \end{bmatrix} \quad (28)$$

therefore, we can get:

$$x(t) = C^T\, W(t) \quad (29)$$
$$x'(t) = \frac{d\,x(t)}{dt} = \frac{d\,C^T\, W(t)}{dt} = C^T\,\frac{d\,W(t)}{dt} \quad (30)$$
$$x'(t) = C^T\, D^{(1)}\, W(t) \quad (31)$$
$$x''(t) = \frac{d\,x'(t)}{dt} = \frac{d\,C^T D^{(1)}\, W(t)}{dt} = C^T\,\frac{d\,D^{(1)}\, W(t)}{dt} \quad (32)$$
$$x''(t) = C^T\, D^{(2)}\, W(t) \quad (33)$$

where,

$$C^T = [\,c_0\,, c_1\,, c_2\,, c_3, c_4] \quad (34)$$
$$W(t) = [\,p_0(t)\,,\ p_1(t)\,,\ p_2(t)\,,\ p_3(t),\ p_4(t)]^T. \quad (35)$$

Thus, we will conclude the following:

- From Eq. 29, we find:

$$x(t) = C^T\, W(t) = [\,c_0\,, c_1\,, c_2\,, c_3, c_4]\,[\,p_0(t)\,,\ p_1(t)\,,\ p_2(t)\,,\ p_3(t),\ p_4(t)]^T \quad (36)$$

and that give:

$$x(t) = c_0\, p_0(t) + c_1\, p_1(t) + c_2\, p_2(t) + c_3\, p_3(t) + c_4\, p_4(t) \quad (37)$$

- From Eq. 31, we find:

$$x'(t) = C^T D^{(1)} W(t)$$
$$= [c_0, c_1, c_2, c_3, c_4]\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 \\ 2 & 0 & 10 & 0 & 0 \\ 0 & 6 & 0 & 14 & 0 \end{bmatrix}\begin{bmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{bmatrix} \quad (38)$$

and that gives:

$$x'(t) = 2c_1p_0(t) + 2c_3p_0(t) + 6c_2p_1(t) + 6c_4p_1(t) + 10c_3p_2(t) + 14c_4p_3(t) \quad (39)$$

- From Eq. 33, we find:

$$x''(t) = C^T D^{(2)} W(t)$$
$$= [c_0, c_1, c_2, c_3, c_4]\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 12 & 0 & 0 & 0 & 0 \\ 0 & 60 & 0 & 0 & 0 \\ 40 & 0 & 140 & 0 & 0 \end{bmatrix}\begin{bmatrix} p_0(t) \\ p_1(t) \\ p_2(t) \\ p_3(t) \\ p_4(t) \end{bmatrix} \quad (40)$$

and that gives:

$$x''(t) = 12c_2p_0(t) + 40c_4p_0(t) + 60c_3p_1(t) + 140c_4p_2(t) \quad (41)$$

Now, by using Eqs. 37, 39, and 41 we can find the residual function $R(t)$ of problem 15 as follows:

$$R(t) = x''(t) - f\big(t, x(t), x'(t)\big) \quad (42)$$
$$R(t) = 12c_2p_0(t) + 40c_4p_0(t) + 60c_3p_1(t) + 140c_4p_2(t) - f\big(t, c_0\, p_0(t) + c_1\, p_1(t) + c_2\, p_2(t) + c_3\, p_3(t) + c_4\, p_4(t), 2c_1p_0(t) + 2c_3p_0(t) + 6c_2p_1(t) + 6c_4p_1(t) + 10c_3p_2(t) + 14c_4p_3(t)\big). \quad (43)$$

Then, we apply Tau method, which can be defined as:

$$\int_0^1 R(t)\, p_r(t)\, dt = 0 \quad ; r = 0,1,2,\dots, m-2. \quad (44)$$

For m =4 , we get:

$$\int_0^1 R(t)\, p_0(t)\, dt = 0 \quad (45)$$
$$\int_0^1 R(t)\, p_1(t)\, dt = 0 \quad (46)$$
$$\int_0^1 R(t)\, p_2(t)\, dt = 0. \quad (47)$$

From Eqs. 45, 46, and 47, we can get three linear or nonlinear equations. In addition, two linear equations can be found by applying the initial conditions (Eq. 16).

Therefore, we will get a system of five linear equations or a system of five non-linear equations, then by solving these equations, we will obtain the constants:

c_0 , c_1 , c_2 , c_3 and c_4.

Through these constants, the approximate-analytical solution of problem 15 can be obtained, which is:

$$x(t) \approx c_0\, p_0(t) + c_1\, p_1(t) + c_2\, p_2(t) + c_3\, p_3(t) + c_4\, p_4(t). \quad (48)$$

Anyone can redo the above description if m is not equal to 4, or if the ODE is a boundary value problem.

4. Applied examples

In this section, we will solve four applied examples. These examples varied between linear and nonlinear first-order ODE and linear and nonlinear second-order ODE. To show the accuracy of the used method, we compute the absolute error:

$$error = |x_{exact}(t) - x_{app}(t)|$$

where, $x_{app}(t)$ is the approximate solution that we found.

Example 1: Consider the first order linear ODE:

$$x'(t) = t^2 - x(t) \quad ; \quad t \in [0,1]$$

with:

$$x(0) = 1$$

Solution: By deriving the equation, we get:

$$x''(t) = 2t - x'(t)$$

with:

$$x(0) = 1 \quad ; \quad x'(0) = -1.$$

From Eq. 48, we describe the approximate solution as:

$$x(t) = c_0\, p_0(t) + c_1\, p_1(t) + c_2\, p_2(t) + c_3\, p_3(t) + c_4\, p_4(t).$$

By applying the Eqs. 9-13, we obtain:

$$x(t) = c_0(1) + c_1(2t-1) + c_2(6t^2 - 6t + 1) + c_3(20t^3 - 30t^2 + 12t - 1) + c_4(70t^4 - 140t^3 + 90t^2 - 20t + 1).$$

Now, we find the residual function $R(t)$:

$$x''(t) + x'(t) - 2t = 0$$
$$R(t) = x''(t) + x'(t) - 2t.$$

By substituting the Eqs. 39 and 41, we obtain:

$$R(t) = 12c_2p_0(t) + 40c_4p_0(t) + 60c_3p_1(t) + 140c_4p_2(t) + 2c_1p_0(t) + 2c_3p_0(t) + 6c_2p_1(t) + 6c_4p_1(t) + 10c_3p_2(t) + 14c_4p_3(t) - 2t.$$

Now, we apply the Eqs. 45, 46, and 47:

- $\int_0^1 R(t)\, p_0(t)\, dt = 0.$

Which simplifies into:

$$2c_1 + 12c_2 + 2c_3 + 40c_4 = 1.$$

- $\int_0^1 R(t)\, p_1(t)\, dt = 0\,.$

Which simplifies into:

$2c_2 + 20c_3 + 2c_4 = \frac{1}{3}$

- $\int_0^1 R(t)\, p_2(t)\, dt = 0.$

Which simplifies into:

$c_3 + 14c_4 = 0.$

Moreover, we apply the initial conditions to get:

$$c_0 - c_1 + c_2 - c_3 + c_4 = 1$$
$$2c_1 - 6c_2 + 12c_3 - 20c_4 = -1.$$

Therefore, we have the following system of linear equations:

$$2c_1 + 12c_2 + 2c_3 + 40c_4 = 1$$
$$2c_2 + 20c_3 + 2c_4 = \frac{1}{3}$$
$$c_3 + 14c_4 = 0$$
$$c_0 - c_1 + c_2 - c_3 + c_4 = 1$$
$$2c_1 - 6c_2 + 12c_3 - 20c_4 = -1.$$

By solving the above system, we get:

$$c_0 = 0.701208981001727$$
$$c_1 = -0.189119170984456$$
$$c_2 = 0.115223291389095$$
$$c_3 = 0.005181347150259$$
$$c_4 = -0.000370096225019.$$

Then, the approximate-analytical solution is:

$$x(t) = 1 - t + (0.502590673575090)t^2 + (0.155440414507840)t^3 - (0.025906735751330)t^4.$$

The exact-analytical solution is:

$$x(t) = t^2 - 2t + 2 - e^{-t}.$$

A numerical solution for this problem can be found in Table 1.

Table 1: Numerical result for example 1

t	$x_{app}(t)$	error
0	1	0
0.000275	0.999725038011652	1.96 e-10
0.000550	0.999450152059538	7.82 e-10
0.000825	0.999175342163047	1.76 e-9
0.001100	0.998900608341568	3.12 e-9
0.001375	0.998625950614484	4.87 e-9
0.001650	0.998351369001174	7.00 e-9
0.001925	0.998076863521017	9.52 e-9
0.002200	0.997802434193383	1.24 e-8
0.002475	0.997528081037643	1.57 e-8
0.002750	0.997253804073162	1.94 e-8

The researcher Islam (2015) solved this problem using the fourth-order Runge-Kutta method for different values of h, where the absolute error ranged between 3.01e-11-10.51e-7.

Example 2: Consider the following first order non-linear ODE:

$$x'(t) = 1 + t^2 - x^2(t) \quad ; \quad t \in [0,4].$$

With:

$$x(0) = 1$$

Solution: By deriving the equation, we get:

$$x''(t) = 2t - 2x(t)x'(t).$$

With:

$$x(0) = 1 \quad ; \quad x'(0) = 0.$$

From Eq. 48, we describe the approximate solution as:

$$x(t) = c_0\, p_0(t) + c_1\, p_1(t) + c_2\, p_2(t) + c_3\, p_3(t) + c_4\, p_4(t).$$

By applying the Eqs. 9-13, we obtain:

$$x(t) = c_0(1) + c_1(2t-1) + c_2(6t^2 - 6t + 1) + c_3(20t^3 - 30t^2 + 12t - 1) + c_4(70t^4 - 140t^3 + 90t^2 - 20t + 1).$$

Now, we find the residual function $R(t)$:

$$x''(t) + 2x(t)x'(t) - 2t = 0$$
$$R(t) = x''(t) + 2x(t)x'(t) - 2t.$$

By substituting the Eqs. 39 and 41, we obtain:

$$R(t) = 12c_2p_0(t) + 40c_4p_0(t) + 60c_3p_1(t) + 140c_4p_2(t) + 2[(2c_1p_0(t) + 2c_3p_0(t) + 6c_2p_1(t) + 6c_4p_1(t) + 10c_3p_2(t) + 14c_4p_3(t))(c_0\, p_0(t) + c_1\, p_1(t) + c_2\, p_2(t) + c_3\, p_3(t) + c_4\, p_4(t))] - 2t.$$

Now, we apply the Eqs. 45, 46, and 47:

- $\int_0^1 R(t)\, p_0(t)\, dt = 0.$

Which simplifies into:

$$2c_0c_1 + 2c_0c_3 + 2c_2c_1 + 2c_4c_1 + 2c_2c_3 + 2c_3c_4 + 6c_2 + 20c_4 - \frac{1}{2} = 0$$

- $\int_0^1 R(t)\, p_1(t)\, dt = 0.$

Which simplifies into:

$$\frac{2}{3}c_1^2 + 2c_0c_2 + 2c_0c_4 + 2c_3c_1 + 2c_4c_1 + 2c_2c_4 + \frac{4}{5}c_2^2 + \frac{6}{7}c_3^2 + \frac{8}{9}c_4^2 + 10c_3 - \frac{1}{6} = 0.$$

- $\int_0^1 R(t)\, p_2(t)\, dt = 0.$

Which simplifies into:

$$2c_0c_3 + \frac{6}{5c_2c_1} + 2c_4c_1 + \frac{52}{35}c_2c_3 + \frac{34}{21}c_3c_4 + 14c_4 = 0$$

Moreover, we apply the initial conditions to get:

$$c_0 - c_1 + c_2 - c_3 + c_4 = 1$$
$$c_1 - 3c_2 + 6c_3 - 10c_4 = 0.$$

Therefore, we have the following system of non-linear equations:

$2c_0c_1 + 2c_0c_3 + 2c_2c_1 + 2c_4c_1 + 2c_2c_3 + 2c_3c_4 + 6c_2 + 20c_4 - \frac{1}{2} = 0$
$\frac{2}{3}{c_1}^2 + 2c_0c_2 + 2c_0c_4 + 2c_3c_1 + 2c_4c_1 + 2c_2c_4 + \frac{4}{5}{c_2}^2 + \frac{6}{7}{c_3}^2 + \frac{8}{9}{c_4}^2 + 10c_3 - \frac{1}{6} = 0$
$2c_0c_3 + \frac{6}{5c_2c_1} + 2c_4c_1 + \frac{52}{35}c_2c_3 + \frac{34}{21}c_3c_4 + 14c_4 = 0$
$c_0 - c_1 + c_2 - c_3 + c_4 = 1$
$c_1 - 3c_2 + 6c_3 - 10c_4 = 0.$

By solving the above system, we get:

$c_0 = 1.057788802033481$
$c_1 = 0.099655382508691$
$c_2 = 0.048805452205811$
$c_3 = 0.005656935799317$
$c_4 = -0.001218935931284.$

Then, the approximate-analytical solution is:

$x(t) = 1 + (0.007750405439796)t^2 + (0.292609746366100)t^3 - (0.089735515189880)t^4.$

The exact-analytical solution is:

$$x(t) = t + \frac{e^{-t^2}}{1+\int_0^t e^{-v^2}dv}.$$

Numerical results for this problem can be found in Table 2.

Table 2: Numerical result for example 2

t	$x_{app}(t)$	error
0	1	0
0. 000017	1.000000000002241	2.24 e-12
0.000034	1.000000000008971	8.96 e-12
0.000051	1.000000000020198	2.02 e-11
0.000068	1.000000000035930	3.58 e-11
0.000085	1.000000000056176	5.60 e-11
0.000102	1.000000000080946	8.06 e-11
0.000119	1.000000000110247	1.10 e-10
0.000136	1.000000000144088	1.43 e-10
0.000153	1.000000000182477	1.81 e-10
0.000170	1.000000000225424	2.24 e-10

The researchers Geng et al. (2009) solved this problem using the piecewise variational iteration method, where the absolute error ranged approximately between 1.00e-10-8.00e-9.

Example 3: Consider the second order linear ODE:

$x''(t) + 4x(t) = sin3t \quad ; \quad t \in [0,2].$

With:

$x(0) = 1 \quad ; \quad x'(0) = 2$

Solution: In the same way that we have used in the previous examples, we find the residual function $R(t)$:

$R(t) = 12c_2p_0(t) + 40c_4p_0(t) + 60c_3p_1(t) + 140c_4p_2(t) + 4c_0\, p_0(t) + 4c_1\, p_1(t) + 4c_2\, p_2(t) + 4\, p_3(t) + 4c_4\, p_4(t) - sin3t.$

Now, we apply the Eqs. 45, 46, and 47 and the initial conditions to get the following system of linear equations:

$c_0 + 3c_2 + 10c_4 = \frac{1}{12} - \frac{1}{12}cos3 = 0$
$c_1 + 15c_3 = \frac{1}{6}sin3 - \frac{1}{4}cos3 - \frac{1}{4} = 0$
$c_2 + 35c_4 = \frac{5}{6}sin3 + \frac{5}{36}cos3 - \frac{5}{36} = 0$
$c_0 - c_1 + c_2 - c_3 + c_4 = 1$
$c_1 - 3c_2 + 6c_3 - 10c_4 = 1.$

By solving the above system, we get:

$c_0 = 1.242589382390501$
$c_1 = -0.141939874229722$
$c_2 = -0.379985319711381$
$c_3 = 0.010863866482100$
$c_4 = 0.006319927939368.$

Then, the approximate-analytical solution is:

$x(t) = 1 + 2t - (2.037034392371466)t^2 - (0.667512578547320)t^3 + (0.442394955755760)t^4.$

The exact-analytical solution is:

$x(t) = cos2t + 1.3sin2t - 0.2sin3t.$

A numerical solution for this problem can be found in Table 3.

The researchers Hossain et al. (2017) solved this problem using Putcher's fifth-order Runge-Kutta method for different values of h, where the absolute error ranged between 4.46e-13-4.18e-7.

Example 4: Consider the second order non-linear ODE:

$x''(t) = -\left(1 + \left(x''(t)\right)^2\right) \quad ; \quad t \in [0,1]$
With: $x(0) = 1 \quad ; \quad x(1) = 0.$

Solution: In the same way that we have used in the previous examples, we obtain the approximate analytical solution**:**

$x(t) = 1 - (0.743668661762932)t - (0.234407900119338)t^2 - (0.043846756112060)t^3 + (0.021923303678080)t^4.$

This problem has no exact solution. Hence, it is clear to us the importance of the used method, as this method provides an accurate approximate solution that can be used as an effective alternative to the exact solution.

5. Conclusion

The approximate solutions that we obtained during this work are accurate solutions and very close to the exact solutions, based on the comparison that we made between our results and the exact solutions. This comparison was based on finding the absolute error and then comparing it with the absolute error calculated by other approximate methods. Hence, the importance of the method becomes clear to us, as this method provides an accurate approximate solution that can be used as an effective alternative to the exact solution if it does not exist. For the next works, one can use this method to solve partial (and fuzzy) differential equations.

Table 3: Numerical result for example 3

t	$x_{app}(t)$	error
0	1	0
0. 000125	1.000249968170034	5.78 e-10
0.000250	1.000499872674922	2.31 e-9
0.000375	1.000749713506846	5.20 e-9
0.000500	1.000999490657990	9.24 e-9
0.000625	1.001249204120541	1.44 e-8
0.000750	1.001498853886687	2.08 e-8
0.000875	1.001748439948621	2.82 e-8
0.001000	1.001997962298538	3.69 e-8
0.001125	1.002247420928633	4.66 e-8
0.001250	1.002496815831106	5.75 e-8

Compliance with ethical standards

Conflict of interest

The author(s) declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

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